STA 302f15 Assignment One[1]

Please do these review questions in preparation for Quiz One and Test One; they are not to be handed in. This material will not directly be on the final exam. The following formulas will be supplied with Quiz One. You may use them without proof.

$E(X) = \sum_x x\, p_X(x)$	$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$
$E(g(X)) = \sum_x g(x)\, p_X(x)$	$E(g(\mathbf{X})) = \sum_{x_1} \cdots \sum_{x_p} g(x_1, \ldots, x_p)\, p_{\mathbf{X}}(x_1, \ldots, x_p)$
$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$	$E(g(\mathbf{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_p)\, f_{\mathbf{X}}(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$
$E(\sum_{i=1}^n a_i X_i) = \sum_{i=1}^n a_i E(X_i)$	$Var(X) = E\left((X - \mu_X)^2 \right)$
$Cov(X,Y) = E\left((X - \mu_X)(Y - \mu_Y) \right)$	$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

1. This question is very elementary, but it may help to clarify some basic concepts. The discrete random variables X and Y have joint distribution

	$X=1$	$X=2$	$X=3$
$Y=2$	2/12	3/12	1/12
$Y=1$	2/12	1/12	3/12

   (a) What is the marginal distribution of X?
   (b) What is the marginal distribution of Y?
   (c) Are X and Y independent? Answer Yes or No and show your work.
   (d) Calculate $E(X)$. Show your work.
   (e) Denote a "centered" version of X by $X_c = X - E(X) = X - \mu_X$.
      i. What is the probability distribution of X_c?
      ii. What is $E(X_c)$? Show your work.
      iii. What is the probability distribution of X_c^2?
      iv. What is $E(X_c^2)$? Show your work.
   (f) What is $Var(X)$? If you have been paying attention, you don't have to show any work.
   (g) Calculate $E(Y)$. Show your work.
   (h) Calculate $Var(Y)$. Show your work. You may use Question 5 if you wish.
   (i) Calculate $Cov(X,Y)$. Show your work.
   (j) Let $Z_1 = g_1(X,Y) = X + Y$. What is the probability distribution of Z_1? Show some work.
   (k) Calculate $E(Z_1)$. Show your work.
   (l) Do we have $E(X+Y) = E(X) + E(Y)$? Answer Yes or No. Note that the answer does not require independence.

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(m) Let $Z_2 = g_2(X, Y) = XY$. What is the probability distribution of Z_2? Show some work.

(n) Calculate $E(Z_2)$. Show your work.

(o) Do we have $E(XY) = E(X)E(Y)$? Answer Yes or No. The connection to independence is established in Question 4.

2. Let X be a discrete random variable and let a be a constant. Using the expression for $E(g(X))$ at the beginning of this assignment, show $E(a) = a$. Is the result still true if X is continuous?

3. Let a be a constant and $Pr\{Y = a\} = 1$. Find $Var(Y)$. Show your work.

4. Let X_1 and X_2 be continuous random variables that are *independent*. Using the expression for $E(g(\mathbf{X}))$ above, show $E(X_1X_2) = E(X_1)E(X_2)$. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence." Because X_1 and X_2 are continuous, you will need to integrate. Does your proof still apply if X_1 and X_2 are discrete?

5. Using the definitions of variance covariance along with the linear property $E(\sum_{i=1}^{n} a_iY_i) = \sum_{i=1}^{n} a_iE(Y_i)$ (no integrals), show the following:

   (a) $Var(Y) = E(Y^2) - \mu_Y^2$

   (b) $Cov(X, Y) = E(XY) - E(X)E(Y)$

   (c) If X and Y are independent, $Cov(X, Y) = 0$. Of course you may use Problem 4.

6. Let X be a random variable and a be a constant. Show

   (a) $Var(aX) = a^2Var(X)$.

   (b) $Var(X + a) = Var(X)$.

7. Show $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$.

8. Let X and Y be random variables, and let a and b be constants. Show $Cov(X + a, Y + b) = Cov(X, Y)$.

9. Let X and Y be random variables, with $E(X) = \mu_x$, $E(Y) = \mu_y$, $Var(X) = \sigma_x^2$, $Var(Y) = \sigma_y^2$, $Cov(X, Y) = \sigma_{xy}$ and $Corr(X, Y) = \rho_{xy}$. Let a and b be non-zero constants.

   (a) Find $Cov(aX, Y)$.

   (b) Find $Corr(aX, Y)$. Do not forget that a could be negative.

10. Let $y_1, \ldots, y_n$ be numbers, and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$. Show

    (a) $\sum_{i=1}^{n}(y_i - \overline{y}) = 0$

    (b) $\sum_{i=1}^{n}(y_i - \overline{y})^2 = \sum_{i=1}^{n} y_i^2 - n\overline{y}^2$

    (c) The sum of squares $Q_m = \sum_{i=1}^{n}(y_i - m)^2$ is minimized when $m = \overline{y}$.

11. Let $Y_1, \ldots, Y_n$ be independent random variables with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2$ for $i = 1, \ldots, n$. For this question, please use definitions and familiar properties of expected value, not integrals.

   (a) Find $E(\sum_{i=1}^n Y_i)$. Are you using independence?

   (b) Find $Var\left(\sum_{i=1}^n Y_i\right)$. What earlier questions are you using in connection with independence?

   (c) Using your answer to the last question, find $Var(\overline{Y})$.

   (d) A statistic T is an *unbiased estimator* of a parameter θ if $E(T) = \theta$. Show that $\overline{Y}$ is an unbiased estimator of μ. This is very quick.

   (e) Let $a_1, \ldots, a_n$ be constants and define the linear combination L by $L = \sum_{i=1}^n a_i Y_i$. What condition on the a_i values makes L an unbiased estimator of μ?

   (f) Is $\overline{Y}$ a special case of L? If so, what are the a_i values?

   (g) What is $Var(L)$?

12. Here is a simple linear regression model. Let $Y = \beta_0 + \beta_1 x + \epsilon$, where β_0 and β_1 are constants (typically unknown), x is a known, observable constant, and ϵ is a random variable with expected value zero and variance σ^2.

   (a) What is $E(Y)$?

   (b) What is $Var(Y)$?

   (c) Suppose that the distribution of ϵ is normal, so that it has density $f(\epsilon) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2\sigma^2}}$. Find the distribution of Y. Show your work. Hint: differentiate the cumulative distribution function of Y.

   (d) Suppose there are two equations:

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 x_1 + \epsilon_1 \\ Y_2 &= \beta_0 + \beta_1 x_2 + \epsilon_2 \end{aligned}$$

   with $E(\epsilon_1) = E(\epsilon_2) = 0$, $Var(\epsilon_1) = Var(\epsilon_2) = \sigma^2$ and $Cov(\epsilon_1, \epsilon_2) = 0$. What is $Cov(Y_1, Y_2)$? Just give the number of the problem you solved earlier.

13. Which statement is true? (Quantities in boldface are matrices of constants.)

    (a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
    (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{BA}+\mathbf{CA}$
    (c) Both a and b
    (d) Neither a nor b

14. Which statement is true?

    (a) $a(\mathbf{B}+\mathbf{C}) = a\mathbf{B}+a\mathbf{C}$
    (b) $a(\mathbf{B}+\mathbf{C}) = \mathbf{B}a+\mathbf{C}a$
    (c) Both a and b
    (d) Neither a nor b

15. Which statement is true?

    (a) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{AB}+\mathbf{AC}$
    (b) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
    (c) Both a and b
    (d) Neither a nor b

16. Which statement is true?

    (a) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$
    (b) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
    (c) Both a and b
    (d) Neither a nor b

17. Which statement is true?

    (a) $\mathbf{A}'' = \mathbf{A}$
    (b) $\mathbf{A}''' = \mathbf{A}'$
    (c) Both a and b
    (d) Neither a nor b

18. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Which statement is true?
    (a) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
    (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    (c) Both a and b
    (d) Neither a nor b
19. Which statement is true?
    (a) $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
    (b) $(\mathbf{A}+\mathbf{B})' = \mathbf{B}' + \mathbf{A}'$
    (c) $(\mathbf{A}+\mathbf{B})' = (\mathbf{B}+\mathbf{A})'$
    (d) All of the above
    (e) None of the above
20. Which statement is true?
    (a) $(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$
    (b) $(a+b)\mathbf{C} = \mathbf{C}a + \mathbf{C}b$
    (c) $(a+b)\mathbf{C} = \mathbf{C}(a+b)$
    (d) All of the above
    (e) None of the above
21. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$? No proof is required here.
22. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.
23. Matrix multiplication does not commute. That is, if $\mathbf{A}$ and $\mathbf{B}$ are matrices, in general it is *not* true that $\mathbf{AB} = \mathbf{BA}$ unless both matrices are 1×1. Establish this important fact by making up a simple numerical example in which $\mathbf{A}$ and $\mathbf{B}$ are both 2×2 matrices. Carry out the multiplication, showing $\mathbf{AB} \neq \mathbf{BA}$. This is also the point of Question 13.
24. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?
