Name Jerry

Student Number ______________________

# STA 302 f2014 Quiz 5A

1. (4 points) Let $\mathbf{Y}_1 \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{Y}_2 \sim N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$ be *independent* multivariate normal random vectors. Find the distribution of $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$. Show your work. You may use the fact that for random vectors (as for scalars), the moment-generating function of a sum is the product of moment-generating functions.

$$M_Y(t) = M_{Y_1}(t)\, M_{Y_2}(t)$$

$$= e^{t'\mu_1 + \frac{1}{2} t'\Sigma_1 t}\, e^{t'\mu_2 + \frac{1}{2} t'\Sigma_2 t}$$

$$= e^{t'(\mu_1+\mu_2) + \frac{1}{2} t'(\Sigma_1+\Sigma_2) t}$$

MGF of $N(\mu_1+\mu_2, \Sigma_1+\Sigma_2)$

2. (6 points) Let $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{I}_p)$, and let $\boldsymbol{\Sigma}$ be a $p \times p$ symmetric non-negative definite matrix with spectral decomposition $\boldsymbol{\Sigma} = \mathbf{CDC}'$. Using *moment-generating functions*, find the distribution of $\mathbf{Y} = \mathbf{CD}^{1/2}\mathbf{Z} + \boldsymbol{\mu}$. Show your work.

Just to be clear, the formula sheet says that if $\mathbf{Y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\mathbf{AY} \sim N_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$. Do *not* use this theorem; use moment-generating functions.

$$M_Y(t) = M_{CD^{1/2}Z+\mu}(t) \overset{\text{Formula Sheet}}{=} e^{t'\mu} M_{CD^{1/2}Z}(t)$$

$$\overset{\text{Formula sheet again}}{=} e^{t'\mu} M_Z((CD^{1/2})'t)$$

$$= e^{t'\mu} e^{\frac{1}{2}((CD^{1/2})'t)'((CD^{1/2})'t)}$$

$$= e^{t'\mu} e^{\frac{1}{2}t'CD^{1/2}D^{1/2'}C't}$$

$$= e^{t'\mu} e^{\frac{1}{2}t'(CDC')t}$$

$$= e^{t'\mu + \frac{1}{2}t'\Sigma t}, \text{ So}$$

$$Y \sim N(\mu, \Sigma)$$

Even though we did not use $\Sigma^{1/2}$. pretty cool huh?

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# STA 302 f2014 Quiz 5B

A model for simple regression through the origin is $Y_i = \beta x_i + \epsilon_i$, where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are independent with expected value 0 and variance $\sigma^2$. In homework, you found that the least squares estimate of $\beta$ is $\widehat{\beta} = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{j=1}^n x_j^2}$, and that $\widehat{\beta}$ is an unbiased estimator of $\beta$.

1. (1 points) The estimator $\widehat{\beta}$ is a linear combination of the $Y_i$ values: $\widehat{\beta} = \sum_{i=1}^n c_i^{(0)} Y_i$. What are the constants $c_i^{(0)}$?

$$c_i^{(0)} = \frac{x_i}{\sum_{j=1}^n x_j^2}$$

2. (3 points) For a *general* linear combination of the form $L = \sum_{i=1}^n c_i Y_i$, what condition on the $c_i$ values makes $L$ an unbiased estimator of $\beta$? ~~Obtain the condition any way you want, and then show that with your condition on the $c_i$ values, $E(L) = \beta$.~~
Show your work. Circle your answer

$$\beta = E(L) = E\left(\sum_{i=1}^n c_i Y_i\right) = \sum_{i=1}^n c_i \, E(Y_i) = \sum_{i=1}^n c_i \beta x_i$$

So make $\sum_{i=1}^n c_i x_i = 1$

3. (6 points) Show that the variance of $L$ is minimized by choosing $c_i = c_i^{(0)}$ for $i = 1, \ldots, n$.

$$Var(L) = Var\left(\sum_{i=1}^{n} c_i Y_i\right) = \sum_{i=1}^{n} c_i^2 \, Var(Y_i)$$

$$= \sum_{i=1}^{n} c_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} c_i^2,$$

and $$\sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} \left(c_i - c_i^{(0)} + c_i^{(0)}\right)^2$$

$$= \sum_{i=1}^{n} \left(c_i - c_i^{(0)}\right)^2 + 2\sum_{i=1}^{n} \left(c_i - c_i^{(0)}\right) c_i^{(0)} + \sum_{i=1}^{n} c_i^{(0)2}$$

Now $$\sum_{i=1}^{n} c_i c_i^{(0)} - \sum_{i=1}^{n} c_i^{(0)2} = \sum_{i=1}^{n} \frac{c_i x_i}{\sum_{j=1}^{n} x_j^2} - \sum_{i=1}^{n} \frac{x_i^2}{\left(\sum_{j=1}^{n} x_j^2\right)^2}$$

$$= \frac{1}{\sum_{j=1}^{n} x_j^2} - \frac{\cancel{\sum_{i=1}^{n} x_i^2}}{\cancel{\sum x_j^2} \sum x_j^2} = 0$$

So $$\sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} \left(c_i - c_i^{(0)}\right)^2 + \frac{1}{\sum_{j=1}^{n} x_j^2}$$

The second term is positive. The first term is non-negative, and equals zero iff $c_i = c_i^{(0)}$ for all $i = 1, \ldots, n$. In this case the variance is minimized.