Name Jerry

Student Number

STA 302 f2014 Quiz 2A

1. (~~5~~ 4 points) Let $Y = X_1 + X_2$, where X_1 and X_2 are independent, $X_1 \sim \chi^2(\nu_1)$ and $Y \sim \chi^2(\nu_1 + \nu_2)$, where ν_1 and ν_2 are both positive. Show $X_2 \sim \chi^2(\nu_2)$.

$$M_Y(t) \overset{ind.}{=} M_{X_1}(t) M_{X_2}(t)$$

$$\Rightarrow (1-2t)^{-\frac{(\nu_1+\nu_2)}{2}} = (1-2t)^{-\nu_1/2} M_{X_2}(t)$$

$$\Rightarrow \cancel{(1-2t)^{-\nu_1/2}} (1-2t)^{-\nu_2/2} = \cancel{(1-2t)^{-\nu_1/2}} M_{X_2}(t)$$

$$\Rightarrow (1-2t)^{-\nu_2/2} = M_{X_2}(t)$$

So $X_2 \sim \chi^2(\nu_2)$

2. ~~(2 points)~~ Let $Y_i = \beta x_i + \epsilon_i$ for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a distribution with expected value zero and variance σ^2, and β and σ^2 are unknown constants. The numbers $x_1, \ldots, x_n$ are known, observed constants. In homework, you derived the least squares estimator $\widehat{\beta} = \frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}$. Just use this; don't derive it again.

For these questions, you may use well known properties of the expected value and variance without proof.

(a) (1 point) What is $E(Y_i)$? Show some work.

$$E(Y_i) = E(\beta x_i + \varepsilon_i) = \beta x_i + E(\varepsilon_i) = \beta x_i + 0$$
$$= \beta x_i$$

(b) (1 point) What is $Var(Y_i)$? Show some work.

$$Var(Y_i) = Var(\beta x_i + \varepsilon_i) = Var(\varepsilon_i) = \sigma^2$$

(c) (2 points) Recall that a statistic is an *unbiased estimator* of a parameter if the expected value of the statistic is equal to the parameter. Is $\widehat{\beta}$ an unbiased estimator of β? Answer Yes or No and show your work.

$$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right) = \frac{1}{\sum_{i=1}^{n} x_i^2} E\left(\sum_{i=1}^{n} x_i Y_i\right)$$

$$= \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i E(Y_i) = \frac{1}{\sum x_i^2} \sum_{i=1}^{n} x_i \beta x_i$$

$$= \beta \frac{\cancel{\sum x_i^2}}{\cancel{\sum x_i^2}} = \beta$$ Yes, unbiased

(d) (2 points) What is $Var(\widehat{\beta})$? Show your work. Circle your answer.

$$Var(\hat{\beta}) = Var\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right) = \frac{1}{(\sum x_i^2)^2} Var\left(\sum_{i=1}^{n} x_i Y_i\right)$$

$$= \frac{1}{(\sum x_i^2)^2} \sum_{i=1}^{n} x_i^2 Var(Y_i) = \frac{\sum_{i=1}^{n} x_i^2 \sigma^2}{(\sum_{i=1}^{n} x_i^2)^2}$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

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STA 302 f2014 Quiz 2B

1. (5 points) Let $X_1, \ldots, X_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. Find the distribution of the sample mean $\overline{X}$. Zero marks for the correct answer without a proof.

$$M_{\bar{X}}(t) = M_{\frac{1}{n}\Sigma X_i}(t) = M_{\Sigma X_i}(t/n)$$

$$= \prod_{i=1}^{n} M_{X_i}(t/n) = \prod_{i=1}^{n} e^{\mu t/n + \frac{1}{2}\sigma^2(t^2/n^2)}$$

$$= e^{n(\mu t/n + \frac{1}{2}\sigma^2 t^2/n^2)}$$

$$= e^{\mu t + \frac{1}{2}(\frac{\sigma^2}{n}) t^2}$$

So $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

2. (5 points) Let $Y_i = \beta x_i + \epsilon_i$ for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a distribution with expected value zero and variance σ^2, and β and σ^2 are unknown constants. The numbers $x_1, \ldots, x_n$ are known, observed constants.

(a) Find the Least Squares estimate of β by minimizing the function

$$Q(\beta) = \sum_{i=1}^{n} (Y_i - \beta x_i)^2$$

over all values of β. Let $\widehat{\beta}$ denote the point at which $Q(\beta)$ is minimal. **Circle your formula for $\widehat{\beta}$.** Don't bother with the second derivative test.

$$\frac{dQ}{d\beta} = \frac{d}{d\beta} \sum_{i=1}^{n} (Y_i - \beta x_i)^2 = \sum_{i=1}^{n} \frac{d}{d\beta} (Y_i - \beta x_i)^2$$

$$= \sum_{i=1}^{n} 2(Y_i - \beta x_i)(-x_i) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i Y_i - \beta x_i^2) = 0 \Rightarrow \sum_{i=1}^{n} x_i Y_i = \beta \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \beta = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2} \quad \text{so} \quad \hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$

(b) Calculate $\widehat{\beta}$ for the following data set. ~~Your~~ The answer is a number. Show some work. Circle your answer.

x^2	9	25	36	16	4
x	3	5	6	4	2
y	-6	-4	-22	12	7
xy	-18	-20	-132	48	14

$\Sigma x^2 = 90$

$\Sigma xy = -108$

$$\hat{\beta} = \frac{-108}{90} = -1.2$$