STA 302f13 Assignment Three[1]

For this assignment, Chapter 2 in the text covers matrix algebra and Chapter 3 covers random vectors. You are responsible for what is in this assignment, not everything that's in the text. Questions 1 and 2 are to be done with R. *Please print the two sets of R output on separate pieces of paper. You may be asked to hand one of them in, but not the other.* Except for the R parts, these problems are preparation for the quiz in tutorial on Friday October 3d, and are not to be handed in.

1. Using R, do textbook question 2.14, just parts (c), (g), (h) and (m). Also do textbook question 2.15, just part (e). Label the parts of your output with comment statements. Bring the printout to the quiz.

2. Using R, do textbook question 2.77. Bring the printout to the quiz.

3. Recall the definition of linear independence. The columns of $\mathbf{X}$ are said to be *linearly dependent* if there exists a $p \times 1$ vector $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{Xv} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{Xv} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$. Let $\mathbf{A}$ be a square matrix. Show that if the columns of $\mathbf{A}$ are linearly dependent, $\mathbf{A}^{-1}$ cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

4. Do problem 2.23 in the text.

5. Let $\mathbf{A}$ be a non-singular square matrix (meaning that the inverse exists). Prove $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$.

6. Using Question 5, prove that the if the inverse of a symmetric matrix exists, it is also symmetric.

7. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

8. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\boldsymbol{\Sigma}$ can be written as $\boldsymbol{\Sigma} = \mathbf{CDC}'$, where $\mathbf{C}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\boldsymbol{\Sigma}$, $\mathbf{D}$ is a diagonal matrix of the corresponding eigenvalues, and $\mathbf{C}'\mathbf{C} = \mathbf{CC}' = \mathbf{I}$.

   (a) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.

      i. What is $\mathbf{D}^{-1}$?

      ii. Show $\boldsymbol{\Sigma}^{-1} = \mathbf{CD}^{-1}\mathbf{C}'$

   (b) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.

      i. What do you think $\mathbf{D}^{1/2}$ might be?

      ii. Define $\boldsymbol{\Sigma}^{1/2}$ as $\mathbf{CD}^{1/2}\mathbf{C}'$. Show $\boldsymbol{\Sigma}^{1/2}$ is symmetric.

      iii. Show $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$.

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(c) Now return to the situation where the eigenvalues of the square symmetric matrix $\boldsymbol{\Sigma}$ are all strictly positive. Define $\boldsymbol{\Sigma}^{-1/2}$ as $\mathbf{C}\mathbf{D}^{-1/2}\mathbf{C}'$, where the elements of the diagonal matrix $\mathbf{D}^{-1/2}$ are the reciprocals of the corresponding elements of $\mathbf{D}^{1/2}$.

   i. Show that the inverse of $\boldsymbol{\Sigma}^{1/2}$ is $\boldsymbol{\Sigma}^{-1/2}$, justifying the notation.

   ii. Show $\boldsymbol{\Sigma}^{-1/2}\boldsymbol{\Sigma}^{-1/2} = \boldsymbol{\Sigma}^{-1}$.

(d) The (square) matrix $\boldsymbol{\Sigma}$ is said to be *positive definite* if $\mathbf{v}'\boldsymbol{\Sigma}\mathbf{v} > 0$ for all vectors $\mathbf{v} \neq \mathbf{0}$. Show that the eigenvalues of a positive definite matrix are all strictly positive. Hint: start with the definition of an eigenvalue and the corresponding eigenvalue: $\boldsymbol{\Sigma}\mathbf{v} = \lambda\mathbf{v}$. This is *much* cleaner than the way I did it in lecture.

(e) Let $\boldsymbol{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\boldsymbol{\Sigma}^{-1}$ exists.

9. Do problem 2.7 in the text.

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10. Do problem 3.9 in the text.

11. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$. Recall the definition $\mathbf{AB} = [\sum_k a_{i,k}b_{k,j}]$.

12. If the $p \times 1$ random vector $\mathbf{X}$ has variance-covariance matrix $\boldsymbol{\Sigma}$ and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{AX}$ is $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$. Start with the definition of a variance-covariance matrix:

$$cov(\mathbf{Z}) = E(\mathbf{Z} - \boldsymbol{\mu}_z)(\mathbf{Z} - \boldsymbol{\mu}_z)'.$$

13. Do problem 3.10 in the text.

14. Let the $p \times 1$ random vector $\mathbf{X}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and let $\mathbf{c}$ be a $p \times 1$ vector of constants. Find $cov(\mathbf{X} + \mathbf{c})$. Show your work.

15. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{Y}$ be a $q \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$. Recall that $C(\mathbf{X}, \mathbf{Y})$ is the $p \times q$ matrix $C(\mathbf{X}, \mathbf{Y}) = E\left((\mathbf{X} - \boldsymbol{\mu}_x)(\mathbf{Y} - \boldsymbol{\mu}_y)'\right)$.

   (a) What is the (i, j) element of $C(\mathbf{X}, \mathbf{Y})$?

   (b) Find an expression for $cov(\mathbf{X} + \mathbf{Y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $C(\mathbf{X}, \mathbf{Y})$. Show your work.

   (c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for all i and j.

   (d) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be a $q \times 1$ vector of constants. Find $C(\mathbf{X} + \mathbf{c}, \mathbf{Y} + \mathbf{d})$. Show your work.

16. Do problem 3.20 in the text. The answer is in the back of the book.

17. Do problem 3.21 in the text. The answer is in the back of the book.
