Family (Last) Name ______

Given (First) Name Jerry

Student Number ______

STA 302s13 Quiz 5A

For reference, the general linear model in matrix form is $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is an $n \times (k+1)$ matrix of observable constants, the columns of $\mathbf{X}$ are linearly independent, $\boldsymbol{\beta}$ is a $(k+1) \times 1$ vector of unknown constants (parameters), and $\boldsymbol{\epsilon}$ is an $n \times 1$ vector of unobservable random variables with $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $cov(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}_n$, where $\sigma^2 > 0$ is an unknown constant parameter. The least squares estimator of $\boldsymbol{\beta}$ is $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$.

1. (1 point) Suppose we want to estimate $\mathbf{a}'\boldsymbol{\beta}$ based on sample data. What estimator is the most natural choice?

$$a'\hat{\beta}$$

2. (2 points) Show that the estimator you have proposed is unbiased. Show the full calculati

Do not use $E(\hat{\beta}) = \beta$ directly.

$$E(a'\hat{\beta}) = a'E(\hat{\beta}) = a'E((X'X)^{-1}X'Y)$$
$$= a'(X'X)^{-1}X'E(Y) = a'(X'X)^{-1}X'X\beta = a'\beta$$

3. (2 points) The natural estimator is a *linear* unbiased estimator of the form $\mathbf{c}_0'\mathbf{Y}$. What is the $n \times 1$ vector $\mathbf{c}_0$? Make sure your answer has the correct dimension. ~~You have more room than you need.~~

Circle your expression for c_0.

$$c_0'Y = a'\hat{\beta} = a'(X'X)^{-1}X'Y, \text{ so}$$

$$c_0' = a'(X'X)^{-1}X', \text{ and}$$

$$c_0 = X(X'X)^{-1}a$$

4. (5 points) As you saw in homework and lecture, the *best* linear unbiased estimator is $\mathbf{c}_0'\mathbf{Y}$. An important part of the proof is to show $(\mathbf{c}-\mathbf{c}_0)'\mathbf{c}_0 = 0$, using the constraint $\mathbf{X}'\mathbf{c} = \mathbf{a}$. Please carry out the calculation.

$$(c - c_0)' c_0 = c' c_0 - c_0' c_0$$

$$= c' X (X'X)^{-1} a - \left(X (X'X)^{-1} a \right)' X (X'X)^{-1} a$$

using $X'c = a \iff c'X = a'$

$$= a' X (X'X)^{-1} a - a' (X'X)^{-1} \underbrace{X'X (X'X)^{-1}}_{I} a$$

$$= 0$$

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STA 302s13 Quiz ~~4~~5B

For reference, $\mathbf{Y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ means $M_{\mathbf{Y}}(\mathbf{t}) = e^{\mathbf{t}'\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}}$. You may use these facts without proof: $M_{\mathbf{AY}}(\mathbf{t}) = M_{\mathbf{Y}}(\mathbf{A}'\mathbf{t})$ and $M_{\mathbf{Y}+\mathbf{c}}(\mathbf{t}) = e^{\mathbf{t}'\mathbf{c}} M_{\mathbf{Y}}(\mathbf{t})$.

1. (~~6~~ points) Let $\mathbf{Z} = \boldsymbol{\Sigma}^{-1/2}(\mathbf{Y} - \boldsymbol{\mu})$. What is the distribution of $\mathbf{Z}$? Show your work. **Circle your final answer.**

$$M_Z(t) = M_{Y-\mu}(\Sigma^{-1/2\,\prime} t) = M_{Y-\mu}(\Sigma^{-1/2} t)$$

$$= e^{-t'\mu} M_Y(\Sigma^{-1/2} t)$$

$$= e^{-t'\mu} e^{t'\mu + \frac{1}{2}(\Sigma^{-1/2} t)' \Sigma (\Sigma^{-1/2} t)}$$

$$= e^{-t'\mu} e^{t'\mu} e^{\frac{1}{2} t' \Sigma^{-1/2} \Sigma \Sigma^{-1/2} t}$$

$$= e^{\frac{1}{2} t' \underbrace{\Sigma^{-1/2} \Sigma^{1/2}}_{I} \underbrace{\Sigma^{1/2} \Sigma^{-1/2}}_{I} t} = e^{\frac{1}{2} t' t}$$

So $Z \sim N_p(0, I)$

2. (~~4~~ 3 points) What is the distribution of $\mathbf{Z}'\mathbf{Z}$? Give reasons for your answer. You may just cite well-known facts about functions of normal random variables.

$$Z'Z = \sum_{j=1}^{p} Z_j^2, \text{ where } Z_j \sim N(0,1),$$

so $Z_j^2 \sim \chi^2(1)$. The Z_j quantities are independent, so this is a sum of independent chi-squares, and $Z'Z \sim \chi^2(p)$