Multivariate Normal (8.1)

See Ch 4

write up & save

$X \sim N(\mu, \sigma^2)$ means $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$M_{\underset{\sim}{A}\underset{\sim}{X}}(\underset{\sim}{t}) = M_{\underset{\sim}{X}}(\underset{\sim}{A}'\underset{\sim}{t})$ $\quad M_{\underset{\sim}{X}+\underset{\sim}{c}}(\underset{\sim}{t}) = e^{\underset{\sim}{t}'\underset{\sim}{c}} M_{\underset{\sim}{X}}(\underset{\sim}{t})$

Leave a space

Let $Z_1, \dots, Z_p \overset{iid}{\sim} N(0,1)$, $Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_p \end{pmatrix}$ $E(Z) = 0$, $\text{cov}(Z) = I_p$

$$M_{\underset{\sim}{Z}}(\underset{\sim}{t}) = \prod_{i=1}^{p} M_{Z_i}(t_i) = \prod_{i=1}^{p} e^{\frac{1}{2}t_i^2} = e^{\frac{1}{2}\sum_{i=1}^{p} t_i^2} = e^{\frac{1}{2}\underset{\sim}{t}'\underset{\sim}{t}}$$

Let Σ be a symmetric non-negative def matrix, & $\mu \in \mathbb{R}^p$

Let $X = \Sigma^{1/2} Z + \mu$ $\quad E(X) = \mu$, $\text{cov}(X) = \Sigma^{1/2} I \Sigma^{1/2\,\prime}$
$= \Sigma^{1/2}\Sigma^{1/2} = \Sigma$

show sym and exam

$$M_{\underset{\sim}{X}}(t) = M_{\Sigma^{1/2}Z+\mu}(t) = e^{t'\mu} M_{\Sigma^{1/2}Z}(\underset{\sim}{t}) = e^{t'\mu} M_Z(\Sigma^{1/2}t)$$

$$= e^{t'\mu} M_Z(\Sigma^{1/2}t) = e^{t'\mu} e^{\frac{1}{2}(\Sigma^{1/2}t)'\Sigma^{1/2}t}$$

$$= e^{t'\mu} e^{\frac{1}{2}t'\Sigma^{1/2}\Sigma^{1/2}t} = e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

Define A multivariate normal X with parameters $\mu = E(X)$ & $\Sigma = \text{cov}(X)$ as one with MGF

$$M_X(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

write in the space

(8.2)

$X \sim N_p(\mu, \Sigma)$ means

$$M_X(t) = e^{t'\mu + \frac{1}{2} t'\Sigma t}$$

- If $p = 1$, $M_X(t) = e^{t\mu + \frac{1}{2} t \sigma^2 t} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, UVN

So UVN is a special case of MVN

- Let $Y = \underset{r \times p}{A} X$ Find $M_Y(\underset{r \times 1}{t})$

$$M_{AX}(t) = M_X(A't)$$
$$= e^{(A't)'\mu + \frac{1}{2}(A't)'\Sigma A't}$$
$$= e^{t'(A\mu) + \frac{1}{2} t'(A\Sigma A')t}$$

MGF of $N_r(A\mu, A\Sigma A')$

- A could be $1 \times p$ – call it a'

is a scalar random variable.

$Y = \underset{\sim}{a}'\underset{\sim}{X}$ is a scalar random variable.

Find $M_Y(t)$

Remember t is 1×1 ($= r$, # of rows), so $t' = t$

$$M_Y(t) = e^{t'(a'\mu) + \frac{1}{2} t'(a'\Sigma a)t}$$
$$= e^{(a'\mu)t + \frac{1}{2}(a'\Sigma a)t^2}$$

$Y \sim N(a'\mu, a'\Sigma a)$ possibly degenerate

So any linear combination of the elements of a multivariate normal is univariate normal.

- In particular $\underset{\sim}{a}$ could have a one in position j & all the rest zeros

Then,

$$\underset{\sim}{a}'\underset{\sim}{X} = (0\ 0 \cdots 1\ 0 \cdots 0)\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_j \\ \vdots \\ X_p \end{pmatrix} = X_j$$

$$E(X_j) = a'\mu = \mu_j$$

$$Var(X_j) = \underbrace{a'\Sigma}_{\text{row } j} a = \sigma_j^2$$

So the one-dimensional marginals are univariate normal.

And $AX \sim N_r(A\mu, A\Sigma A')$ means the r-dimensional marginals are multivariate normal, just picking out the means, variances & covariances.

It's a lot easier than integrating

8.4

An easy example
If you do it the easy way

Let $\mathbf{X} = (X_1, X_2, X_3)'$ be multivariate normal with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the joint distribution of Y_1 and Y_2.

In matrix terms

$Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$ means $\mathbf{Y} = \mathbf{AX}$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$\mathbf{Y} = \mathbf{AX} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$ $\quad A\mu = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$$A\Sigma A' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 6 \end{bmatrix}$$

FOR THE MULTIVARIATE NORMAL, ZERO covariance Implies independence

For any random variables, independence implies zero covariance.

For MVN, it goes in the other direction too.

Suppose $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$. Then

$$t'\Sigma t = (t_1 \;\; t_2) \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$= (t_1 \sigma_1^2 , \; t_2 \sigma_2^2) \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$= \sigma_1^2 t_1^2 + \sigma_2^2 t_2$$

Now let $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_0^2 \end{pmatrix}\right]$

Have $X_1 \sim N(\mu_1, \sigma_1^2)$

$X_2 \sim N(\mu_2, \sigma_2^2)$

JOINT MGF

$$M_X(t) = e^{t'\mu + \frac{1}{2} t' \Sigma t}$$

$$= e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2)}$$

$$= e^{\mu_1 t_1} e^{\mu_2 t_2} e^{\frac{1}{2}\sigma_1^2 t_1^2} e^{\frac{1}{2}\sigma_2^2 t_2^2}$$

$$= e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2} e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2}$$

$$= M_{X_1}(t_1)\, M_{X_2}(t_2)$$ INDEPENDENT

And these same calculations apply to partitioned matrices so it's true of multivariate normals too.

Theorem

pos def this time

$$X \sim N_p(\mu, \Sigma) \Rightarrow (X-\mu)' \Sigma^{-1} (X-\mu) \sim \chi^2(p)$$

Proof $Y = X - \mu \sim N_p(0, \Sigma)$

$$Z = \Sigma^{-1/2} Y \sim N_p(\Sigma^{-1/2} 0, \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}\,'})$$

$$= N_p(0, \Sigma^{-1/2}\Sigma^{1/2}\Sigma^{1/2}\Sigma^{-\frac{1}{2}}) = N_p(0, I_p)$$

so $Z_1 \cdots Z_p$ are ind st norm

Then

$$(X-\mu)' \Sigma^{-1} (X-\mu) = Y' \Sigma^{-1} Y$$

$$= (\Sigma^{1/2} Z)' \Sigma^{-1} \Sigma^{1/2} Z$$

$$= Z' \underbrace{\Sigma^{1/2} \Sigma^{-1/2}}_{I} \underbrace{\Sigma^{-1/2} \Sigma^{1/2}}_{I} Z$$

$$= Z' Z$$

$$= \sum_{i=1}^{p} Z_i^2 \sim \chi^2(p)$$

done

8.8

Independence of $\bar{X}$ & S^2 ← write formulas

Let $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, i.e.

$$\underset{\sim}{X} = (X_1 \cdots X_n)' \sim N_n(\underset{\sim}{1}\mu, \sigma^2 I_n)$$

Let
$$\underset{\sim}{Y} = \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \\ \bar{X} \end{pmatrix}_{(n+1)\times 1} = \underset{\sim}{A}\,\underset{\sim}{X}$$

$$= \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & & -\frac{1}{n} \\ & & & & \\ & & & -\frac{1}{n} & 1-\frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

$Y \sim N(A\mu, A\Sigma A')$

Note $\quad a'Y = 0 \quad$ for $a' = (1\,1 \cdots 1\,0)$

$$Y = \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \\ \hline \bar{X} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{Y}_1 \\ \hline \underset{\sim}{Y}_2 \end{pmatrix}$$

$\mathrm{Cov}(\underset{\sim}{Y}_1, \underset{\sim}{Y}_2) = 0$
Implies independence

Show $\operatorname{Cov}(\bar{X}, (X_j - \bar{X})) = 0$ (8.9)

$$\operatorname{Cov}(\bar{X}, (X_j - \bar{X})) = E(\bar{X}(X_j - \bar{X})) - \underbrace{E(\ )E(\ )}_{0}$$

$$= E\left(X_j \frac{1}{n}\sum_{i=1}^{n} X_i\right) - E(\bar{X}^2)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i X_j) - \left(\operatorname{Var}(\bar{X}) + (E(\bar{X}))^2\right)$$

$$= \frac{1}{n}\left(EX_j^2 + \sum_{i \neq j}^{n-1} E(X_i)E(X_j)\right) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \frac{1}{n}\left(\sigma^2 + \mu^2 + (n-1)\mu^2\right) - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{\sigma^2}{n} + \frac{(1+n-1)\mu^2}{n} - \frac{\sigma^2}{n} - \mu^2 = 0$$

So $\underset{\sim}{Y}_1 = \begin{pmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}$ & $Y_2 = \bar{X}$ are independent

Thus $\bar{X}$ & $g(\underset{\sim}{Y}_1) = \dfrac{\sum (X_i - \bar{X})^2}{n-1}$ are independent

$\boxtimes$

See STH 256 text

8.10

Theorem

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

Proof

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)$$

$\parallel$

$Y \sim \chi^2(n)$

$$= \frac{1}{\sigma^2}\left(\Sigma(X_i-\bar{X})^2 + 2(\bar{X}-\mu)\,\Sigma(X_i-\bar{X}) + n(\bar{X}-\mu)^2\right)$$

$$= \frac{(n-1)S^2}{\sigma^2} + 0 + \underbrace{\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2(1)}$$

Hence

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \qquad \blacksquare$$

t - Distribution

Recall if $Z \sim N(0,1)$ & $W \sim \chi^2(r)$ are independent, then

$$T = \frac{Z}{\sqrt{W/r}} \sim t(r)$$

This is a definition

Save

Now $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, so

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$$

And $W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ ind of Z,

So

$$T = \frac{\sqrt{n}(\bar{X} - \mu)/\sigma}{\sqrt{\frac{(n-1)S^2}{\sigma^2} \Big/ (n-1)}} = \frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t(n-1)$$

Basis of the usual confidence intervals and tests, And useful as a warmup.