Properties of the Least-Squares Estimators

We can do it, but is it any good?

$\hat{\underset{\sim}{\beta}} \neq \underset{\sim}{\beta}$ obviously

$\hat{\underset{\sim}{\beta}}$ is random, has a probability distribution

$\hat{\beta}_j$

β_j

Unbiased means $E(\hat{\theta}) = \theta \quad (\forall\, \theta \in \Theta)$

$$E(\hat{\beta}) = E\{(X'X)^{-1}X'Y\} = (X'X)^{-1}X'E\{Y\}$$
$$= (X'X)^{-1}X'X\beta = \beta$$

So $E(\hat{\beta}_j) = \beta_j$ for $j = 0, \ldots, k$

How about the variance? Smaller variance means more accurate estimation.

Notice $\hat{\beta} = (X'X)^{-1}X'Y$ means

$\hat{\beta}_j$ is a linear combination of the Y values

Row j of $A = (X'X)^{-1}X'$ times Y

Gauss-Markov theorem says $\hat{\beta}$ is

BLUE

Gauss - Markov

Let $E(Y) = X\beta$ & $\mathrm{cov}(Y) = \sigma^2 I_n$, and $\hat{\beta} = (X'X)^{-1}X'Y$

Then $a'\hat{\beta}$ has the smallest variance of any linear unbiased estimator of the form $c'Y$ with $E(c'Y) = a'\beta$.
In particular if $a_j = 1$ and $a_i = 0$ for all $i \neq j$, $\mathrm{Var}(\hat{\beta}_j)$ is less than the variance of any other linear unbiased estimator of β_j.

Proof $\quad E(c'Y) = c'E(Y) = \underset{1\times n}{c'}\,\underset{n\times(k+1)}{X}\,\underset{(k+1)\times 1}{\beta} = a'\beta \quad \forall\, \beta$ one ele

$$\Rightarrow c'X = a' \Leftrightarrow a = X'c$$

$$\mathrm{Var}(c'Y) = c'\,\mathrm{cov}(Y)\,c = c'\sigma^2 I_n c = \sigma^2 c'c$$

Seek $c \in \mathbb{R}^n$ that makes this smallest subject to $X'c = a$.

What "should" c be? $a'\hat{\beta} = (X'X)^{-1}X'Y$ should be $c'Y$

So try $c' = a'(X'X)^{-1}X' \Leftrightarrow c = X(X'X)^{-1}a$. Add & subtract

$$c'c = [\underbrace{c - X(X'X)^{-1}a}_{A} + \underbrace{X(X'X)^{-1}a}_{B}]'[\underbrace{c - X(X'X)^{-1}a}_{A} + \underbrace{X(X'X)^{-1}a}_{B}]$$

$$= A'A + A'B + B'A + B'B$$

Look at $A'B$, substituting $a = X'c$

$$A'B = [c - X(X'X)^{-1}X'c]' X(X'X)^{-1}X'c$$

$$= c'X(X'X)^{-1}X'c - c'X(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}X'c = 0$$

So $B'A = 0' = 0$ also

In $B'B$, dont substitute

$$B'B = a'(X'X)^{-1}\underbrace{X'X(X'X)^{-1}}_{I}a = \underset{\text{Free of } c}{a'(X'X)^{-1}a}$$

So $c'c = A'A$ plus a positive term free of c

$$A'A = \underset{1 \times n}{[c - X(X'X)^{-1}a]'} \underset{n \times 1}{[c - X(X'X)^{-1}a]}$$

It's a sum of squares, ≥ 0, & $= 0$ only if all terms are zero.

$$c - X(X'X)^{-1}a = 0 \iff c = X(X'X)^{-1}a$$

Thus $a'\hat{\beta}$ is BLUE of $a'\beta$

$\blacksquare$