

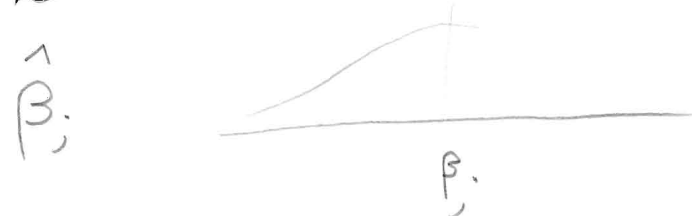
# Properties of the Least-Squares Estimators

7.1

We can do it, but is it any good?

$\hat{\beta} \neq \beta$  obviously

$\hat{\beta}$  is random, has a probability distribution



Unbiased means  $E(\hat{\theta}) = \theta$  ( $\forall \theta \in \Theta$ )

$$\begin{aligned} E(\hat{\beta}) &= E\{(X'X)^{-1}X'Y\} = (X'X)^{-1}X'E\{Y\} \\ &= (X'X)^{-1}X'X\beta = \beta \end{aligned}$$

so  $E(\hat{\beta}_j) = \beta_j$  for  $j=0, \dots, k$

How about the variance? Smaller variance means more accurate estimation.

Notice  $\hat{\beta} = (X'X)^{-1}X'Y$  means

$\hat{\beta}_j$  is a linear combination of the  $Y$ -values

Row  $j$  of  $A = (X'X)^{-1}X'$  times  $Y$

Gauss-Markov theorem says  $\hat{\beta}$  is

BLUE

## Gauss - Markov

7.2

Let  $E(Y) = X\beta$  &  $\text{cov}(Y) = \sigma^2 I_n$ , and  $\hat{\beta} = (X'X)^{-1}X'Y$

Then  $a'\hat{\beta}$  has the smallest variance of any linear unbiased estimator of the form  $c'Y$  with  $E(c'Y) = a'\beta$

In particular if  $a_i = 1$  and  $a_j = 0$  for all  $i \neq j$ ,

$\text{Var}(\hat{\beta}_i)$  is less than the variance of any other linear unbiased estimator of  $\beta_i$ .

Proof  $E(c'Y) = c'E(Y) = c'X\beta = a'\beta \quad \forall \beta$  one  
ele

$$\Rightarrow c'X = a' \Leftrightarrow a = X'c$$

$$\text{Var}(c'Y) = c'\text{cov}(Y)c = c'\sigma^2 I_n c = \sigma^2 c'c$$

Seek  $c \in \mathbb{R}^n$  that makes this smallest subject to  $X'c = a$ .

What "should"  $c$  be?  $a'\hat{\beta} = (X'X)^{-1}X'Y$  should be  $c'Y$

So try  $c' = a'(X'X)^{-1}X' \Leftrightarrow c = X(X'X)^{-1}a$ . Add & subtract

$$c'c = \underbrace{[c - X(X'X)^{-1}a]}_A + \underbrace{X(X'X)^{-1}a}_B \quad X' \quad \underbrace{[c - X(X'X)^{-1}a]}_A + \underbrace{X(X'X)^{-1}a}_B$$

$$= A'A + A'B + B'A + B'B$$

