STA 302f13 Assignment One[1]

The following formulas will be supplied with Quiz One.

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx, \quad E(g(\mathbf{X})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_p)\, f_{\mathbf{X}}(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$$

$$Var(X) = E[(X - \mu_X)^2] \qquad Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

1. Let X be a continuous random variable and let a be a constant. Using the expression for $E(g(X))$ above, show $E(a) = a$.

2. Let X_1 and X_2 be continuous random variables. Using the expression for $E(g(\mathbf{X}))$ above, show $E(X_1 + X_2) = E(X_1) + E(X_2)$. If you assume independence, you get a zero.

3. Let Y_1 and Y_2 be continuous random variables that are *independent*. Using the expression for $E(g(\mathbf{Y}))$, show $E(Y_1 Y_2) = E(Y_1)E(Y_2)$. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence."

4. Using the definitions of variance covariance along with familiar properties of expected value (no integrals), show the following:
   (a) $Var(Y) = E(Y^2) - \mu_Y^2$
   (b) $Cov(X,Y) = E(XY) - E(X)E(Y)$
   (c) If X and Y are independent, $Cov(X,Y) = 0$. Of course you may use Problem 3.

5. In the following, X and Y are random variables, while a and b are fixed constants. For each pair of statements below, one is true and one is false (that is, not true in general). State which one is true, and prove it. Zero marks if you prove both statements are true, even if one of the proofs is correct. Use definitions and familiar properties of expected value, not integrals.
   (a) $Var(aX) = aVar(X)$ or $Var(aX) = a^2Var(X)$
   (b) $Var(aX + b) = a^2Var(X) + b^2$ or $Var(aX + b) = a^2Var(X)$
   (c) $Var(a) = 0$ or $Var(a) = a^2$
   (d) $Cov(X + a, Y + b) = Cov(X,Y) + ab$ or $Cov(X + a, Y + b) = Cov(X,Y)$
   (e) $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$ or $Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$

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6. Let $Y_1, \ldots, Y_n$ be numbers, and $\overline{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$. Show

   (a) $\sum_{i=1}^n (Y_i - \overline{Y}) = 0$

   (b) $\sum_{i=1}^n (Y_i - \overline{Y})^2 = \sum_{i=1}^n Y_i^2 \; - \; n\overline{Y}^2$

   (c) The sum of squares $Q_m = \sum_{i=1}^n (Y_i - m)^2$ is minimized when $m = \overline{Y}$.

7. Let $Y_1, \ldots, Y_n$ be independent random variables with $E(Y_i) = \mu$ and $Var(Y_i) = \sigma^2$ for $i = 1, \ldots, n$. For this question, please use definitions and familiar properties of expected value, not integrals.

   (a) Find $E(\sum_{i=1}^n Y_i)$.

   (b) Find $Var\left(\sum_{i=1}^n Y_i\right)$. Show your work. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence."

   (c) Using your answer to the last question, find $Var(\overline{Y})$.

   (d) A statistic T is an *unbiased estimator* of a parameter θ if $E(T) = \theta$. Show that $\overline{Y}$ is an unbiased estimator of μ. This is very quick.

   (e) Let $a_1, \ldots, a_n$ be constants and define the linear combination L by $L = \sum_{i=1}^n a_i X_i$. What condition on the a_i values makes L an unbiased estimator of μ?

   (f) Is $\overline{Y}$ a special case of L? If so, what are the a_i values?

   (g) What is $Var(L)$?

8. Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance σ^2, so that $T = \frac{\sqrt{n}(\overline{Y}-\mu)}{S} \sim t(n-1)$. This is something you don't need to prove, for now.

   (a) Derive a $(1-\alpha)100\%$ confidence interval for μ. "Derive" means show all the high school algebra. Use the symbol $t_{\alpha/2}$ for the number satisfying $Pr(T > t_{\alpha/2}) = \alpha/2$.

   (b) A random sample with $n = 23$ yields $\overline{Y} = 2.57$ and a sample variance of $S^2 = 5.85$. Using the critical value $t_{0.025} = 2.07$, give a 95% confidence interval for μ. The answer is a pair of numbers.

   (c) Test $H_0 : \mu = 3$ at $\alpha = 0.05$.

      i. Give the value of the T statistic. The answer is a number.

      ii. State whether you reject H_0, Yes or No.

      iii. Can you conclude that μ is different from 3? Answer Yes or No.

      iv. If the answer is Yes, state whether $\mu > 3$ or $\mu < 3$. Pick one.

   (d) Show that using a t-test, $H_0 : \mu = \mu_0$ is rejected at significance level α if and only the $(1-\alpha)100\%$ confidence interval for μ does not include μ_0. The problem is easier if you start by writing the set of $Y_1, \ldots, Y_n$ values for which H_0 is *not* rejected.

(e) In Question 8b, does this mean $Pr\{1.53 < \mu < 3.61\} = 0.95$? Answer Yes or No and briefly explain.

9. In *Linear models in statistics*, do problems 2.3, 2.4 and 2.14 parts a, b, and h. Review Chapter 2 or your linear algebra text as necessary. The answers are in the back of the book. The *trace* of a square matrix $\mathbf{A}$, denoted $tr(\mathbf{A})$, is the sum of the diagonal elements.

10. Which statement is true? (Quantities in boldface are matrices of constants.)

    (a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
    (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{BA}+\mathbf{CA}$
    (c) Both a and b
    (d) Neither a nor b

11. Which statement is true?

    (a) $a(\mathbf{B}+\mathbf{C}) = a\mathbf{B}+a\mathbf{C}$
    (b) $a(\mathbf{B}+\mathbf{C}) = \mathbf{B}a+\mathbf{C}a$
    (c) Both a and b
    (d) Neither a nor b

12. Which statement is true?

    (a) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{AB}+\mathbf{AC}$
    (b) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
    (c) Both a and b
    (d) Neither a nor b

13. Which statement is true?

    (a) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$
    (b) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
    (c) Both a and b
    (d) Neither a nor b

14. Which statement is true?

    (a) $\mathbf{A}'' = \mathbf{A}$
    (b) $\mathbf{A}''' = \mathbf{A}'$
    (c) Both a and b
    (d) Neither a nor b

15. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Which statement is true?
    (a) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
    (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    (c) Both a and b
    (d) Neither a nor b

16. Which statement is true?
    (a) $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
    (b) $(\mathbf{A}+\mathbf{B})' = \mathbf{B}' + \mathbf{A}'$
    (c) $(\mathbf{A}+\mathbf{B})' = (\mathbf{B}+\mathbf{A})'$
    (d) All of the above
    (e) None of the above

17. Which statement is true?
    (a) $(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$
    (b) $(a+b)\mathbf{C} = \mathbf{C}a + \mathbf{C}b$
    (c) $(a+b)\mathbf{C} = \mathbf{C}(a+b)$
    (d) All of the above
    (e) None of the above

18. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$? No proof is required here.

19. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.

20. Recall that an inverse of the matrix $\mathbf{A}$ (denoted $\mathbf{A}^{-1}$) is defined by two properties: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Prove that inverses are unique, as follows. Let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$. Show that $\mathbf{B} = \mathbf{C}$.

21. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?
