University of Toronto in Mississauga

April Examinations 2005
STA 261S
Duration - 3 hours

Aids allowed: Calculator. Formula sheet and tables will be supplied.

1. (3 Points) For a random sample from a Gamma distribution with unknown parameter $\alpha > 0$ and known $\beta = 1$, give a *one-dimensional* statistic that is sufficient for α. Circle the statistic.

2. (5 Points) Let $\widehat{\Theta}_1, \widehat{\Theta}_2, \ldots$ be a sequence of *unbiased* estimators of a parameter θ.

   Prove that if $\lim_{n\to\infty} Var(\widehat{\Theta}_n) = 0$, then $\widehat{\Theta}_n$ is consistent for θ.

3. The formula sheet will be helpful on this question. Let $X_1, \ldots, X_n$ be independent Exponential random variables with parameter θ. Let $\widehat{\Theta}_n = nY_1 = n\min(X_1, \ldots, X_n)$.

   (a) (2 Points) Prove that $\widehat{\Theta}_n$ is also exponentially distributed with parameter θ; start by obtaining the density of Y_1 (use the formula sheet). After that, use moment-generating functions. Show your work.

   (b) (1 Point) Is $\widehat{\Theta}_n$ unbiased? Answer Yes or No and say why. You don't need to show a calculation here.

   (c) (2 Points) Is $\widehat{\Theta}_n$ sufficient? Answer Yes or No and say why you think so. A rigorous proof is not required, but you do need to show a calculation and give a reason for your opinion.

   (d) (5 Points) Is $\widehat{\Theta}_n$ consistent? Answer Yes or No and prove it *using the definition of consistency.*

   (e) (2 Points) Another estimator of θ is $\overline{X}_n$. Find the relative efficiency of $\overline{X}_n$ compared to $\widehat{\Theta}_n$. Give a (very simple) formula for the relative efficiency, and *clearly state which estimator is more efficient.*

4. (5 Points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x;\theta) = \theta x^{\theta-1} I(0 < x < 1)$. Find a Method of Moments estimator of θ. Show your work and *circle your final answer.*

5. (5 Points) We draw a random sample of size $n = 150$ from a Geometric distribution with parameter θ, and observe a sample mean of $\overline{X} = 8.2$. Find a point estimate and an approximate 95% confidence interval for θ. Please give three numbers

   (a) The point estimate.

   (b) The lower limit of the confidence interval.

   (c) The upper limit of the confidence interval.

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6. (10 Points) Let $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, for $i = 1, \ldots, n$, where

- $x_1, \ldots, x_n$ are fixed, known constants
- $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed Normal random variables with $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2$. You cannot observe these error terms directly, so they are not explicitly part of the likelihood function.
- β_0, β_1 and σ^2 are unknown parameters.

Using the fact that Y_i has a normal distribution with $E(Y_i) = \beta_0 + \beta_1 x_i$ and $Var(Y_i) = \sigma^2$, find the Maximum Likelihood Estimator of β_1. Show your work.

You will need to maximize the likelihood function with respect to β_0 as well as β_1, but your goal is to estimate β_1. *Please circle your final answer, the formula for* $\widehat{\beta}_1$.

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x; \alpha) = \frac{\alpha}{x^{\alpha+1}} I(x > 1)$, where $\alpha > 0$.

   (a) (2 Points) Find the density of $Y_i = \ln X_i$. What is the name of the distribution? It's on the formula sheet.

   (b) (2 Points) Prove that $W = 2\alpha \sum_{i=1}^{n} \ln X_i$ has a chisquare distribution with $\nu = 2n$ degrees of freedom.

   (c) (2 Points) Starting with the last item, derive an exact $(1-\alpha)100\%$ confidence interval for α.

8. Let $Y_i = x_i + \epsilon_i$, for $i = 1, \ldots, n$, where

- $x_1, \ldots, x_n$ are fixed, known constants
- $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed Normal$(0,\sigma^2)$ random variables; the parameter σ^2 is unknown. You cannot observe these error terms directly, so they are not explicitly part of the likelihood function.

Recall from the midterm that $Y_i \sim N(x_i, \sigma^2)$. It follows that $W = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - x_i)^2$ has a chisquare distribution with n degrees of freedom. You may use this without proof.

   (a) (7 Points) Use the Neyman-Pearson Lemma to find a most powerful size α test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 < \sigma_0^2$. Denote the critical region of this test by C.

   (b) (2 Points) Why is the test C *uniformly* most powerful for testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 < \sigma_0^2$?

   (c) (5 Points) Find the power function $P_{\sigma^2}\{\mathbf{X} \in C\} = \pi(\sigma^2)$.

   (d) (2 Points) Is the power function increasing, or is it decreasing? Prove it.

   (e) (5 Points) Consider the test of the composite $H_0 : \sigma^2 \geq \sigma_0^2$ against the composite $H_1 : \sigma^2 < \sigma_0^2$.

      i. Draw a rough sketch showing Θ, Θ_0, Θ_1 and the power function.

      ii. Does your picture show that the test C is also size α for testing the composite null hypothesis? Answer Yes or No.

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9. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with expected value μ and variance σ^2. You know several facts that you may use without proof in this question:

   - $\overline{X}$ is normal.
   - $\overline{X}$ and S^2 are independent.
   - If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.
   - If Z is standard normal, then Z^2 is chisquare with one degree of freedom.
   - The sum of independent chisquares is also chisquare; add the degrees of freedom.

   Now the question:

   (a) (8 Points) Show that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.

   (b) (5 Points) Show that $\frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t(n-1)$.

10. (10 Points) Again, let $X_1, \ldots, X_n$ be a random sample from a normal distribution with expected value μ and variance σ^2. Derive an exact size α likelihood ratio test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. In addition to material from earlier in the exam, you may use these facts without proof:

    - The unrestricted MLE is $\widehat{\theta} = (\overline{x}, \widehat{\sigma}^2)$, where $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \overline{x})^2$.
    - The restricted MLE of σ^2 is $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$.

11. Students in a typing class are given educational software for practice at home. As an experiment, they are given one of two programs, at random. After one month of using the programs, they take a standard typing test. The number of mistakes on this test is known to have a Poisson distribution. Fifty students received software program A; their mean number of errors was 18.0. Forty-five students received software program B; their mean number of errors was 20.8. You want to test the null hypothesis that the programs are equally effective versus the alternative that they differ in their effectiveness. You may use the fact that for a Poisson distribution, the Maximum Likelihood Estimate is the sample mean.

    (a) (5 Points) Simplify the Likelihood Ratio λ (see formula sheet) until it equals $\frac{\overline{m}^{(\Sigma x_i + \Sigma y_i)}}{(\overline{x}^{\Sigma x_i})(\overline{y}^{\Sigma y_i})}$, where $\overline{m}$ is the mean of all the observations in both samples.

    (b) (2 Points) Give the value of the chisquare statistic. Your answer is a number. Show your work.

    (c) (1 Point) What is the critical value for this test at $\alpha = 0.05$? The answer is a number.

    (d) (2 Points) Do you reject the null hypothesis? Answer Yes or No. Which program would you recommend in practice? Your answers to this part must be consistent with each other and with your answers to the other parts of the question.

Total marks = 100 points