STA 261s2005 Assignment 6

Do this assignment in preparation for the quiz on Wednesday, March 8th. The questions are practice for the quiz, and are not to be handed in.

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. The "Modified" Central Limit Theorem says that the usual Central Limit Theorem still holds if σ is replaced by any consistent estimator – call it $\widehat{\sigma}$. A Method of Moments estimator of σ will always be consistent. Using the Modified Central Limit Theorem, derive an approximate $(1-\alpha)100\%$ confidence interval for μ. Show your work.

2. A random sample of size $n = 150$ yields a sample mean of $\overline{X} = 8.2$. Give a point estimate and an approximate 95% confidence interval

   (a) For λ, if $X_1, \ldots, X_n$ are from a Poisson distribution with parameter λ. My answer is a point estimate of 8.2, and a confidence interval from 7.24 to 8.66.

   (b) For θ, if $X_1, \ldots, X_n$ are from an Exponential distribution with parameter θ. My answer is a point estimate of 8.2, and a confidence interval from 6.91 to 9.49.

   (c) For μ, if $X_1, \ldots, X_n$ are from a Normal distribution with mean μ and variance one. (This confidence interval is exact, not an approximation.) My answer is a point estimate of 8.2, and a confidence interval from 8.04 to 8.36.

   (d) For θ, if $X_1, \ldots, X_n$ are from a Uniform distribution on $[0, \theta]$. My answer is a point estimate of 16.4, and a confidence interval from 14.88 to 17.92.

   (e) For θ, if $X_1, \ldots, X_n$ are from a Uniform distribution on $[\theta, \theta+1]$. My answer is a point estimate of 7.7, and a confidence interval from 7.65 to 7.75.

   (f) For θ, if $X_1, \ldots, X_n$ are from a Geometric distribution with parameter θ. My answer is a point estimate of 0.122, and a confidence interval from 0.106 to 0.143. Hint: first obtain the Method of Moments estimator $\widehat{\sigma} = \sqrt{\overline{X}(\overline{X}-1)}$.

   (g) For θ, if $X_1, \ldots, X_n$ are from a Binomial distribution with parameters 10 and θ. My answer is a point estimate of 0.82, and a confidence interval from 0.746 to 0.844.

3. Let $X \sim N(\mu, \sigma^2)$. Find the distribution of $Z = \frac{X-\mu}{\sigma}$. Show your work.

4. Let $Z \sim N(0,1)$, and let $Y = Z^2$. Find the distribution of Y. Show your work.

5. Let $Y_1, \ldots, Y_n$ be independent chi-square random variables with respective degree of freedom parameters $\nu_1, \ldots, \nu_n$. Find the distribution of $W = \sum_{i=1}^n Y_i$. Show your work.

6. Let $X_1, \ldots, X_n$ be independent $N(\mu_i, \sigma_i^2)$. What is the distribution of

$$Y = \sum_{i=1}^n \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2 ?$$

   At this point, you should be able to just look at the expression for Y, and know the answer without writing anything. This is where you need to be with the normal distribution.

7. Let $X_1, \ldots, X_n$ be a random sample from an Exponential distribution with parameter θ.

   (a) Derive an *exact* $(1-\alpha)100\%$ confidence interval for θ. Show your work.

   (b) Customers are arriving at a jewelry store according to a stationary Poisson process, which implies that the inter-arrival times are independent exponential random variables. We observe the following times (in minutes) between customer arrivals:

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   7.38   3.50 11.68   3.92   1.21   2.62
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   Give a point estimate and a 95% confidence interval for the expected inter-arrival time. Your answer is three numbers. My answer is a point estimate of 5.05, and a confidence interval from 2.60 to 13.76.

   (c) The *rate* of a Poisson process is $\lambda = 1/\theta$. That is, it is the reciprocal of the expected inter-arrival time. Give a point estimate and a 95% confidence interval for the rate λ (per minute). Use the data from Question 7b. Again, your answer is three numbers. My answer is a point estimate of 0.19 per minute, with a confidence interval from 0.073 to 0.385.

8. In Question 2d, you obtained an approximate, large-sample confidence interval for the parameter of a Uniform $(0, \theta)$ distribution. Can we do better based on the Maximum Likelihood estimator Y_n?

   (a) Try $[Y_n, aY_n]$. Your job is to find the constant a.

   (b) We observe the following random sample from a Uniform$(0, \theta)$ distribution:

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   1.21 2.85 1.65 6.64 3.83 0.35
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   Give a point estimate and an *exact* 95% confidence interval for θ. Your answer is three numbers. Please make your point estimate unbiased. My answer is a point estimate of 7.75, and a confidence interval from 6.64 to 10.94.

9. Try to find something like the last confidence interval for a Shifted Exponential – that is,

$$f(x; \theta) = e^{-(x-\theta)} I(x \geq \theta),$$

   where θ can be any real number.

10. Okay, that didn't work; do you see why?

   (a) Now try to derive a confidence interval of the form $[Y_1 - a, Y_1]$, where $a > 0$. Remember, we always have $Y_1 \geq \theta$.

   (b) We observe the following random sample from a Shifted Exponential with parameter θ:

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   2.05, 2.12, 2.07, 2.64
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   Give a point estimate and an *exact* 95% confidence interval for θ. Your answer is three numbers. Please make your point estimate unbiased. My answer is a point estimate of 1.8, and a confidence interval from 1.3 to 2.05.