STA 261s2006 Assignment 13

Do this assignment in preparation for the Final Exam. The questions are practice for the exam, and are not to be handed in.

1. Let $X_1, \ldots, X_{150}$ be a random sample from a Geometric distribution with parameter θ. We observe a sample mean of $\overline{X} = 2.4$. Using a large sample likelihood ratio statistic, test $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta \neq \frac{1}{2}$. Give the value of the test statistic and the critical value (both numbers), and state whether or not you reject the null hypothesis at $\alpha = 0.05$.

   I get a chisquare of 10.05. The formal test just leads you to the conclusion that $\theta \neq \frac{1}{2}$, but in practice you would want to judge whether $\theta > \frac{1}{2}$, or $\theta < \frac{1}{2}$. Which of these two conclusions do the data suggest?

2. Let $X_1, \ldots, X_{n_1}$ be a random sample from a $N(\mu_1, \sigma_1^2)$ distribution, and let $Y_1, \ldots, Y_{n_2}$ be a random sample from a $N(\mu_2, \sigma_2^2)$ distribution. Using the following data, test $H_0 : \mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ versus $H_1 : \mu_1 \neq \mu_2$, or $\sigma_1^2 \neq \sigma_2^2$, or both. Use a large-sample likelihood ratio test. Give the value of the test statistic and the critical value (both of these are numbers), and state whether or not you reject the null hypothesis at $\alpha = 0.05$.

   X: `13.9 8.0 9.5 9.0 10.9 9.0 9.7 9.7 9.0 12.8 13.3 8.8 12.3 10.6 13.0 8.1 9.8 9.9 11.7 9.9 12.3`

   Y: `10.0 9.9 15.7 11.9 14.9 14.5 12.0 11.1 9.2 11.3 12.9 8.6 11.1 12.0 11.0 13.0 11.1 13.9 10.1 8.9 10.7 11.4 8.5 12.2 12.2 12.1 12.1 15.5 10.7 5.4 13.4 14.2 16.7 14.0 10.6 10.3 8.9 14.4 12.3 13.7 9.3 15.0 7.1 9.7 9.2`

   I get $\chi^2 = 5.8591$. Not quite big enough.

3. Do Exercise 12.43.

4. Do Exercise 12.44. They must mean Exercise 12.21, not 12.32. Notice that they are directing us to use an approximate chi-square test when an exact one is available. You did the exact one in Assignment 12. This time, do the large-sample test. Give the numerical value of the chisquare test statistic, the critical value, and state whether or not you reject the null hypothesis. I get $\chi^2 = 7.85$.

5. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval from zero to θ. Consider two estimators: $\widehat{\Theta}_1 = \frac{n+1}{n} Y_n$ and $\widehat{\Theta}_2 = 2\overline{X}_n$.

   (a) One of these estimators is a Method of Moments estimator, and the other is based on a Maximum likelihood estimator. Which is which?

   (b) Prove that both estimators are unbiased.

   (c) Prove that both estimators are consistent.

   (d) Which estimator is more efficient? Show your work.

   (e) One of the estimators is sufficient, and the other is not. Which one is sufficient?

   (f) Consider $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, and a test that rejects H_0 if $Y_n > \theta_0$.

      i. What is the size of the test?

      ii. Give the power function $\pi(\theta)$. Your answer should apply to all $\theta > 0$.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density

$$f(x; \theta) = \frac{\theta}{x^3} e^{-\theta/x} I(x > 0).$$

   (a) Find a Method of Moments estimator for θ.

   (b) Find the Maximum Likelihood estimator for θ.

   (c) Show that the MLE is biased, but asymptotically it's unbiased. Start by finding the distribution of $Y_i = 1/X_i$.

   (d) Derive an exact $(1 - \alpha)100\%$ confidence interval for θ. You will use critical values of a chisquare distribution with $2n$ degrees of freedom.

   (e) Using the Neyman-Pearson lemma, derive an exact size α test of $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$. The critical value will refer to a chisquare distribution with $2n$ degrees of freedom.

   (f) By the Neyman-Pearson lemma, this test is most powerful for the simple null against the simple alternative. How do you know that it is *uniformly* most powerful for $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$?

   (g) Express the power function of the test in terms of the cumulative distribution function of a chi-square random variable. That is, $\pi(\theta) =$ what?

   (h) Show that the test is size α for testing the composite null hypothesis $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

   (i) Let D be another size size α test of the composite null hypothesis against the composite alternative. How do you know that if $\theta > \theta_0$, the probability of rejecting H_0 is greater with C than with D?

   (j) Starting with a likelihood ratio test and then making a convenient choice of critical values right at the end, give an exact size α test for $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

   (k) I'm working on an expanded version of this question that gives you a small set of data, and asks you to calculate everything for the data and get numerical answers.