STA 261s2006 Assignment 11

Do this assignment in preparation for the quiz on Wednesday, March 29th. The questions are practice for the quiz, and are not to be handed in.

1. Please read Section 12.4 on the Neyman-Pearson lemma. You are not responsible for the proof. Then do exercises 12.10 through 12.15. For 12.12, use the Central Limit Theorem to give a critical region that is *approximately* size α for n large.

2. Show that a critical region based on the Neyman-Pearson lemma will always be defined in terms of the value of a sufficient statistic.

3. Let C be a most powerful critical region of size α for testing the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1$. Let $\theta_0 \in \omega$, and $P_\theta\{(X_1, \ldots, X_n) \in C\} \leq P_{\theta_0}\{(X_1, \ldots, X_n) \in C\}$ for all $\theta \in \omega$. Show that C is also the most powerful critical region of size α for testing the *composite* null hypothesis $H_0 : \theta \in \omega$ against the simple alternative $H_1 : \theta = \theta_1$.

4. Let $X_1, \ldots, X_{n_1}$ be a random sample from a distribution with density

$$f(x; \tau) = \sqrt{\frac{\tau}{2\pi}} e^{-\frac{\tau}{2}x^2}, \text{ where } \tau > 0.$$

   (a) Let $Y = \tau \sum_{i=1}^n X_i^2$. What is the distribution of Y?

   (b) Consider the null hypothesis $H_0 : \tau = \tau_0$ against $H_1 : \tau = \tau_1 > \tau_0$. Show that the most powerful size α critical region can be written as $C = \{x_1, \ldots, x_n : \tau_0 \sum_{i=1}^n x_i^2 < \chi^2_{1-\alpha,n}\}$. This example is noteworthy because the critical region points in the *opposite* direction to the alternative hypothesis.

   (c) Now consider $H_0 : \tau = \tau_0$ against $H_1 : \tau > \tau_0$. Why do you know that C is *uniformly* most powerful for this situation?

   (d) Find the power function $\pi(\tau) = P_\tau(\mathbf{X} \in C)$.

   (e) Is this function increasing, or is it decreasing? Prove it.

   (f) Finally, consider $H_0 : \tau \leq \tau_0$ against $H_1 : \tau > \tau_0$. Draw a rough sketch of Ω, ω, ω' and $\pi(\theta)$. Why does your picture show that the test C is size α for the composite null hypothesis?

   (g) Let D be another size α test of the composite null versus the composite alternative. Show $P_\theta(\mathbf{X} \in D) \leq P_\theta(\mathbf{X} \in C)$ for all $\theta \in \omega'$.

5. Look at Exercise 12.9, except that now the sample size is n. We still want to test $H_0 : \theta = 1$ against $H_1 : \theta = 2$

   (a) Show $\prod_{i=1}^{n} X_i$ is sufficient for θ.

   (b) Show $\sum_{i=1}^{n} - \ln X_i$ is also sufficient for θ.

   (c) Find the distribution of $-\ln X_i$. Show your work.

   (d) What is the distribution of $-\sum_{i=1}^{n} \ln X_i$?

   (e) What is the distribution of $-2\theta \sum_{i=1}^{n} \ln X_i$?

   (f) Show that the most powerful size α critical region can be written as $C = \{x_1, \ldots, x_n : -2\sum_{i=1}^{n} \ln(x_i) < \chi^2_{1-\alpha,2n}\}$. Again, the critical region points away from the alternative hypothesis.