STA 261s2005 Assignment 6

Do this assignment in preparation for the quiz on Wednesday, March 2nd. The questions are practice for the quiz, and are not to be handed in.

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. The "Modified" Central Limit Theorem (see Supplement and Formula Sheet) says that the usual Central Limit Theorem still holds if σ is replaced by any consistent estimator – call it $\widehat{\sigma}$. Using this result, derive an approximate $(1-\alpha)100\%$ confidence interval for μ. Show your work.

2. A random sample of size $n = 150$ yields a sample mean of $\overline{X} = 8.2$. Give a point estimate and an approximate 95% confidence interval

   (a) For λ, if $X_1, \ldots, X_n$ are from a Poisson distribution with parameter λ.

   (b) For θ, if $X_1, \ldots, X_n$ are from an Exponential distribution with parameter θ.

   (c) For μ, if $X_1, \ldots, X_n$ are from a Normal distribution with mean μ and variance one. (This confidence interval is exact, not an approximation.)

   (d) For θ, if $X_1, \ldots, X_n$ are from a Uniform distribution on $[0, \theta]$.

   (e) For θ, if $X_1, \ldots, X_n$ are from a Uniform distribution on $[\theta, \theta + 1]$.

   (f) For θ, if $X_1, \ldots, X_n$ are from a Geometric distribution with parameter θ.

   (g) For θ, if $X_1, \ldots, X_n$ are from a Binomial distribution with parameters 10 and θ.

   In each case, your answer is three numbers.

3. Any linear combination of normal random variables is normal, but it is easier to show with independent normals. We will use the following in lecture, very shortly. Let $X_1, \ldots, X_n$ be independent, $X_i \sim N(\mu_i . \sigma_i^2)$, for $i = 1, \ldots, n$, and let $Y = a_0 + a_1 X_1 + \ldots, a_n X_n$, where $a_0, \ldots, a_n$ are constants. Find the distribution of Y. Show your work.

4. Let $X \sim N(\mu, \sigma^2)$. Find the distribution of $Z = \frac{X-\mu}{\sigma}$. Show your work.

5. Let $Z \sim N(0, 1)$, and let $X = Z^2$. Find the distribution of X. Show your work.

6. Let $X_1, \ldots, X_n$ be independent chi-square random variables with respective degree of freedom parameters $\nu_1, \ldots, \nu_n$. Find the distribution of $Y = \sum_{i=1}^n X_i$. Show your work.

7. Let $X_1, \ldots, X_n$ be independent $N(\mu_i, \sigma_i^2)$. What is the distribution of

$$Y = \sum_{i=1}^{n} \left(\frac{X_i - \mu_i}{\sigma_i} \right)?$$

   At this point, you should be able to just look at the expression for Y, and know the answer without writing anything. This is where you need to be with the normal distribution.

8. Let $X_1, \ldots, X_n$ be a random sample from an Exponential distribution with parameter θ.

   (a) Derive an *exact* $(1-\alpha)100\%$ confidence interval for θ. Show your work.

   (b) Customers are arriving at a jewelry store according to a stationary Poisson process, which implies that the inter-arrival times are independent exponential random variables. We observe the following times (in minutes) between customer arrivals:

   ```
   7.38  3.50 11.68  3.92  1.21  2.62
   ```

   Give a point estimate and a 95% confidence interval for the expected inter-arrival time. Your answer is three numbers.

   (c) The *rate* of a Poisson process is $\lambda = 1/\theta$. That is, it is the reciprocal of the expected inter-arrival time. Give a point estimate and a 95% confidence interval for the rate λ (per minute). Use the data from Question 8b. Again, your answer is three numbers.

   (d) Re-do Question 2b, but this time give an exact confidence interval. Compare the two intervals; they should be close since the sample size is large. Oh no! The chisquare table does not have the critical values you need. Can you find normal approximations of the values? Hint: See Question 6, and use the Central Limit Theorem.

9. We observe the following random sample from a $N(0, \sigma^2)$ distribution:

   ```
   2.99 -0.87 -1.27  0.22 -2.19  4.24  1.98  1.19 -1.08 -1.34
   ```

   Give a point estimate and an *exact* 95% confidence interval for σ^2. Show all your work. Please give an *unbiased* point estimate, not the ordinary sample variance. For the confidence interval, start by finding the distribution of

   $$\frac{1}{\sigma^2} \sum_{i=1}^{n} X_i^2.$$

   As usual, your answer is three numbers.

10. In Question 2d, you obtained an approximate, large-sample confidence interval for the parameter of a Uniform $(0, \theta)$ distribution. Can we do better based on the Maximum Likelihood estimator Y_n?

    (a) Try $[Y_n, aY_n]$. Your job is to find the constant a.

    (b) We observe the following random sample from a Uniform$(0, \theta)$ distribution:

    ```
    1.21 2.85 1.65 6.64 3.83 0.35
    ```

    Give a point estimate and an *exact* 95% confidence interval for θ. Your answer is three numbers. Please make your point estimate unbiased.

11. Try to find something like the last confidence interval for a shifted exponential – that is,

$$f(x; \theta) = e^{-(x-\theta)} I(x \geq \theta)$$

12. Okay, that didn't work; do you see why? Now try a confidence interval of the form $[Y_1 - a, Y_1]$.

13. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x; \lambda) = \lambda x^{\lambda-1} I(0 < x < 1)$, where $\lambda > 0$.

    (a) Derive an exact confidence interval based on the maximum likelihood estimator. Show your work. Hint: Once you obtain the MLE, it may look hopeless, but don't give up. Find the distribution of $W_i = -\ln X_i$. The result should be encouraging.

    (b) Give an *unbiased* point estimator as well as a confidence interval. Hint: The first estimator you might think of is $1/\overline{W}_n$. You can modify this estimator to get one that is unbiased. To do so, you need to find the expected value. You have already found that $\overline{W}_n$ has a Gamma distribution. Accordingly, you can find the expected value you need by directly evaluating

$$E\left[\frac{1}{\overline{W}_n}\right] = \int_0^\infty \frac{1}{w} f_{\overline{W}_n}(w)\, dw.$$