STA 261s2005 Assignment 5.5

Here are some extra problems you should do to prepare for the Midterm on Wednesday, Feb. 23d. The questions are practice for the test (and for the final exam). They are not to be handed in.

1. Use the factorization theorem to show that the Maximum Likelihood Estimate can depend on the sample data only through the value(s) of a sufficient statistic.

2. Let $Y_i = \beta x_i + \epsilon_i$, for $i = 1, \ldots, n$, where

   $x_1, \ldots, x_n$ are fixed, known constants

   are independent and identically distributed Normal(0,σ^2) random variables; the parameters β and σ^2 are unknown.

   (a) Find a Method of Moments estimator of β. Show your work.
      i. Is it unbiased?
      ii. What is the variance of the estimator?
      iii. Is the estimator consistent? (If yes, under what conditions?)

   (b) Find a Method of Moments estimator of σ^2; show your work. Is it unbiased?

   (c) Find the Maximum Likelihood Estimate of β; show your work.
      i. What is the variance of the estimator?
      ii. Is the estimator consistent? (If yes, under what conditions?)

   (d) Find the Maximum Likelihood Estimator of $\widehat{\sigma}^2$; show your work. Is it unbiased?

   (e) Are $\widehat{\beta}$ and $\widehat{\sigma}^2$ jointly sufficient?

   (f) Let L be a linear unbiased estimator of β. That is, $L = \sum_{i=1}^n a_i Y_i$, where $a_1, \ldots, a_n$ are constants, and $E(L) = \beta$. Prove $Var(\widehat{\beta}) \leq Var(L)$.

3. This regression model has an intercept. Let $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, for $i = 1, \ldots, n$, where

   $x_1, \ldots, x_n$ are fixed, known constants

   $\epsilon_1, \ldots, \epsilon_n$ are independent and identically distributed Normal(0,σ^2) random variables; the parameters β_0, β_1 and σ^2 are unknown.

   (a) Find the Maximum Likelihood Estimates of β_0, β_1 and σ^2; show your work.

   (b) Starting with your answer to the question above, show that the Maximum Likelihood Estimate of β_1 can be written

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^n (x_i - \overline{x})^2} = \frac{\sum_{i=1}^n (x_i - \overline{x}) y_i}{\sum_{i=1}^n (x_i - \overline{x})^2}$$

(c) Find $E(\widehat{\beta}_1)$. Is $\widehat{\beta}_1$ unbiased?

(d) Find $E(\widehat{\beta}_0)$. Is $\widehat{\beta}_0$ unbiased?

(e) Find $Var(\widehat{\beta}_1)$.

(f) Is $\widehat{\beta}_1$ consistent? (If yes, under what conditions?)

(g) Are $\widehat{\beta_0}$, $\widehat{\beta_1}$ and $\widehat{\sigma}^2$ jointly sufficient for the parameters β_0, β_1 and σ^2? Answer Yes or No and show your work.