# STA 261s2005 Assignment 11

Do this assignment in preparation for the quiz on Wednesday, April 6th. The questions are practice for the quiz, and are not to be handed in.

1. Let $X_1, \ldots, X_n$ be a random sample from a Normal$(\mu, \sigma^2)$ distribution where $\sigma^2$ is *known*. Derive an exact likelihood ratio test of $H_0 : \mu = \mu_0$ versus the alternative that $\mu \neq \mu_0$. Show your work.

2. Do Exercise 12.20. By $X$, they mean $\sum_{i=1}^n X_i$, where the $X_i$s are Bernoulli with parameter $\theta$. By "symmetry," they mean symmetry about the point $\frac{n}{2}$; show that $f(\frac{n}{2} - a) = f(\frac{n}{2} + a)$.

   I would like to add a part (d). Using the Central Limit Theorem (or the normal approximation to the binomial if you wish, which is what our authors would do), write the critical region explicitly so that it has approximate size $\alpha$ when the sample size is large. Don't bother with any continuity correction.

3. Do Exercise 12.21, but keep going. Add part (c): Now take logs, and study the function $g(x) = \ln x - \frac{x}{\theta_0}$. Using this, show that the critical region can be expressed as
$$C = \left\{ \mathbf{x} \in \mathcal{X} : \frac{2}{\theta_0} \sum_{i=1}^n X_i \leq a \text{ or } \frac{2}{\theta_0} \sum_{i=1}^n X_i \geq b \right\}.$$
   You don't have to specify $a$ and $b$; it's a hard problem. In practice, people would likely use the $\alpha/2$ and $1 - \alpha/2$ critical values of a chisquare distribution with $2n$ degrees of freedom. This, like the usual two-sided test for the variance based on a normal sample, would not exactly be a likelihood ratio test. You could say it's "inspired" by the likelihood ratio test, or something.

4. Do Exercise 12.22. We did it in class a somewhat different way, also acceptable.

5. Let $X_1, \ldots, X_n$ be a random sample from a continuous uniform distribution on $(0, \theta]$. Derive an exact size $\alpha$ likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

6. Let $X_1, \ldots, X_{150}$ be a random sample from a Geometric distribution with parameter $\theta$. We observe a sample mean of $\overline{X} = 2.4$. Using a large sample likelihood ratio statistic, test $H_0 : \theta = \frac{1}{2}$ versus $H_1 : \theta \neq \frac{1}{2}$. Give the value of the test statistic and the critical value (both numbers), and state whether or not you reject the null hypothesis at $\alpha = 0.05$.

   I get a chisquare of 10.05. The formal test just leads you to the conclusion that $\theta \neq \frac{1}{2}$, but in practice you would want to judge whether $\theta > \frac{1}{2}$, or $\theta < \frac{1}{2}$. Which of these two conclusions do the data suggest?

7. Do Exercise 12.23. Instead of the textbook's hint, try L'Hôpital's rule.

8. Do Exercise 12.24. Notice that they are asking only for the *statistic*, not the critical region. The critical region involves slippery $a$ and $b$ values as in Question 3. And in their hint, they mean Example 10.18, not 10.17.

9. Do Exercises 12.25 and 12.26.

10. Let $X_1, \ldots, X_{n_1}$ be a random sample from a $N(\mu_1, \sigma^2)$ distribution, and let $Y_1, \ldots, Y_{n_2}$ be a random sample from a $N(\mu_2, \sigma^2)$ distribution. These are *independent* random samples, meaning that the $X$ and $Y$ values are independent. Using the facts that $\overline{X}$ and $S^2$ are independent and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ for each sample, derive an *exact* likelihood ratio test for $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$. This is the common two-sample $t$-test.

11. Again, we have two independent random samples from normal distributions with the same variance, but this time $\sigma^2$ is *known*. And again, we want to test $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ with an exact likelihood ratio test. Writing the critical region in terms of a statistic that has a standard normal distribution under $H_0$, observe that this critical region has exactly the same form as the one for Question 10, except that the estimator $S_p$ is replaced by the known quantity $\sigma$ and of course we use $z_{\alpha/2}$ instead of $t_{\alpha/2}$.

12. Let $X_1, \ldots, X_{n_1}$ be a random sample from a $N(\mu_1, \sigma_1^2)$ distribution, and let $Y_1, \ldots, Y_{n_2}$ be a random sample from a $N(\mu_2, \sigma_2^2)$ distribution. Using the following data, test $H_0 : \mu_1 = \mu_2$ and $\sigma_1^2 = \sigma_2^2$ versus $H_1 : \mu_1 \neq \mu_2$, or $\sigma_1^2 \neq \sigma_2^2$, or both. Use a large-sample likelihood ratio test. Give the value of the test statistic and the critical value (both of these are numbers), and state whether or not you reject the null hypothesis at $\alpha = 0.05$.

    **X**: `13.9 8.0 9.5 9.0 10.9 9.0 9.7 9.7 9.0 12.8 13.3 8.8 12.3 10.6 13.0 8.1 9.8 9.9 11.7 9.9 12.3`

    **Y**: `10.0 9.9 15.7 11.9 14.9 14.5 12.0 11.1 9.2 11.3 12.9 8.6 11.1 12.0 11.0 13.0 11.1 13.9 10.1 8.9 10.7 11.4 8.5 12.2 12.2 12.1 12.1 15.5 10.7 5.4 13.4 14.2 16.7 14.0 10.6 10.3 8.9 14.4 12.3 13.7 9.3 15.0 7.1 9.7 9.2`

    I get $\chi^2 = 5.8591$. Not quite big enough.

13. Do Exercise 12.43.

14. Let $X_1, \ldots, X_n$ be a random sample from a continuous uniform distribution on $(0, \theta]$. Derive an exact size $\alpha$ likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

15. Do Exercise 12.44. They must mean Exercise 12.21, not 12.32. Notice that they are directing us to use an approximate chi-square test when an exact one is available. This way they avoid those ugly $a$ and $b$ values. Anyway,

    (a) For the large-sample test, give the numerical value of the chisquare test statistic, the critical value, and state whether or not you reject the null hypothesis. I get $\chi^2 = 7.85$.

    (b) Use the test of Question 3 (the one with equal tail areas) to test the same hypothesis. Give the numerical value of the chisquare test statistic, the critical values, and state whether or not you reject the null hypothesis. Oops; the critcal values you need are not in the table: $\chi^2_{0.975,40} = 24.433$, and $\chi^2_{0.025,40} = 59.342$. My chisquare value is $\frac{2}{\theta_0}\sum_{i=1}^n X_i = 70.53$.