STA 261s2005 Assignment 1

Do this assignment in preparation for the quiz on Friday, The questions are practice for the quiz, and are not to be handed in.

1. Here are some review questions about ordinary limits.

   (a) $\lim_{n \to \infty} \frac{5n^2 + 2n + 8}{10n^2 + \sqrt{n} + 1}$

   (b) Let $0 < a < 1$. Derive a convenient formula for $\sum_{k=r}^{\infty} a^k = \lim_{n \to \infty} \sum_{k=r}^{n} a^k$. This is the geometric series. A proof should be in your calculus text if you get stuck.

   (c) $\lim_{n \to \infty} n \sin(1/n)$

   (d) $\lim_{n \to \infty} (1 + \frac{t}{n})^n$

   (e) $\lim_{n \to \infty} (a + \frac{t}{n})^n$

2. Let X have a beta distribution with parameters $\alpha > 0$ and $\beta = 1$. That is,

   $$f_X(x) = \alpha x^{\alpha - 1} I\{0 < x < 1\},$$

   where $\alpha > 0$. Let $Y = 1/X$. Find the density of Y, including an indicator for the support. For what values of k does $E(Y^k)$ exist?

3. Let X have a continuous uniform distribution on (0,1); that is,

   $$f_X(x) = I\{0 < x < 1\}.$$

   Let $Y = -\theta \ln X$, where $\theta > 0$. Find the density of Y, including an indicator for the support. Simplify! This is a familiar density; what is its name?

4. Let X_1 and X_2 be independent random variables from the continuous uniform distribution on (0,1), and let $Y_1 = \frac{X_1 + X_2}{2}$.

   (a) Find the density of Y_1, including an indicator for the support.

   (b) Integrate your answer. Is it a density?

5. Let X_1 and X_2 be independent standard normal random variables, and let $Y_1 = \frac{X_1}{X_2}$. Find the density of Y_1. Your answer should be the *Cauchy* density of Exercise 6.6 on Page 207, with $\alpha = 0$ and $\beta = 1$.

6. Here is some review of moment-generating functions.

   (a) Let $M_X(t)$ be the moment-generating function of the random variable X, and let a be a constant. Show $M_{aX}(t) = M_X(at)$.

   (b) Let $X_1, \ldots, X_n$ be independent random variables with respective moment-generatng functions $M_{X_1}(t), \ldots, M_{X_n}(t)$. Let $Y = \sum_{i=1}^n X_i$. Show $M_Y(t) = \prod_{i=1}^n M_{X_i}(t)$. Be clear about where you use independence.

(c) Derive the moment-generating function of a gamma random variable with parameters α and β. Use your answer to find the mean and variance.

(d) Derive the moment-generating function of X, a geometric random variable with parameter θ. You will need your answer from Question 1b. Use the moment-generating function to find the mean and variance of X.

(e) Let $X_1, \ldots, X_n$ be independent Exponential(θ) random variables. Find the distribution of $\overline{X}_n$.

(f) Let $Z \sim N(0, 1)$, and let $X = Z^2$. Find the distribution of X using moment-generating functions. The integral is surprisingly easy.

(g) Let $X_1, \ldots, X_n$ be independent chi-square random variables with respective degree of freedom parameters $\nu_1, \ldots, \nu_n$. Find the distribution of $Y = \sum_{i=1}^{n} X_i$.

7. Let g be a non-negative function. That is $g(x) \geq 0$ for all x. For any constant a, show that $E(g(X)) \geq aPr\{g(X) \geq a\}$

   (a) For X discrete.

   (b) For X continuous.

   This is *Markov's inequality* (see lecture notes).

8. Read Section 4.4, Pages 141-143. Do exercises 4.29, 4.30, 4.31, 4.32, 4.72, 4.73, 4.74.

9. Let $X_1, \ldots, X_n$ be independent random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$.

   (a) Compute the expected value and variance of $\overline{X}_n$.

   (b) Use Chebyshev's inequality to show that for any constant $c > 0$,

   $$\lim_{n \to \infty} Pr\{|\overline{X}_n - \mu| \geq c\} = 0$$

   This is the *Law of Large Numbers.*