Large-sample Likelihood Ratio Tests

$$G_n^2 = -2 \ln \lambda(\underset{\sim}{X}_n)$$

$$= -2 \ln \frac{L(\hat{\Theta}_0, \underset{\sim}{X})}{L(\hat{\Theta}, \underset{\sim}{X})} \xrightarrow{d} Y \sim \chi^2(\quad)$$

of = signs in H_0

Note H_0 can't have inequalities – a tough unsolved problem.

<u>Ex</u> $X_1 \cdots X_n \overset{iid}{\sim}$ Bernoulli(θ)

$H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

Already have

$$Z_1 = \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}} \;\; \& \;\; Z_2 = \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}}$$

square & get $\chi^2(1)$

Have $\hat{\theta} = \bar{X}_n$, $\hat{\theta}_0 = \theta_0$

2

$$\lambda(\underset{\sim}{x}) = \frac{\prod_{i=1}^{n} \hat{\theta}_0^{\,x_i}(1-\hat{\theta}_0)^{1-x_i}}{\prod_{i=1}^{n} \hat{\theta}^{\,x_i}(1-\hat{\theta})^{1-x_i}} = \frac{\prod_{i=1}^{n} \theta_0^{\,x_i}(1-\theta_0)^{1-x_i}}{\prod_{i=1}^{n} \bar{x}^{\,x_i}(1-\bar{x})^{1-x_i}}$$

$$= \frac{\theta_0^{\Sigma x_i}(1-\theta_0)^{n-\Sigma x_i}}{\bar{x}^{\Sigma x_i}(1-\bar{x})^{n-\Sigma x_i}} = \frac{\theta_0^{n\bar{x}}(1-\theta_0)^{n(1-\bar{x})}}{\bar{x}^{n\bar{x}}(1-\bar{x})^{n(1-\bar{x})}}$$

$$= \left(\frac{\theta_0^{\bar{x}}(1-\theta_0)^{1-\bar{x}}}{\bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}}} \right)^n$$

$$G_n^2 = -2 \ln \lambda(\underset{\sim}{X}) = -2n \ln \left(\frac{\theta_0^{\bar{X}_n}(1-\theta_0)^{1-\bar{X}_n}}{\bar{X}_n^{\bar{X}_n}(1-\bar{X}_n)^{1-\bar{X}_n}} \right)$$

$$= -2n\Big(\bar{X}_n \ln \theta_0 + (1-\bar{X}_n)\ln(1-\theta_0) - \bar{X}_n \ln \bar{X}_n - (1-\bar{X}_n)\ln(1-\bar{X}_n) \Big)$$

$$= 2n\Big[\bar{X}(\ln \bar{X}_n - \ln \theta_0) + (1-\bar{X}_n)\big(\ln(1-\bar{X}_n) - \ln(1-\theta_0)\big) \Big]$$

If 60 out of 100 choose new coffee, (3)

$H_0: \theta = \frac{1}{2}$

$$G^2 = 2 * 100 * \Big[0.6(\ln 0.6 - \ln 0.5)$$
$$+ 0.4(\ln 0.4 - \ln 0.5)\Big]$$

$= 4.03$ > Critical value at $\alpha = 0.05$
$3.84 = 1.96^2$

For Comparison,

$$Z_1^2 = \left(\frac{\sqrt{n}\,(\bar{X}_n - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}}\right)^2 = 4$$

$$Z_2^2 = \left(\frac{\sqrt{n}\,(\bar{X}_n - \theta_0)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}}\right)^2 = 4.17$$