Thursday March 19

More likelihood ratio tests

Ex $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$

$H_0: \sigma^2 \le \sigma_0^2$ vs $H_1: \sigma^2 > \sigma_0^2$

$\Omega = \{(\mu, \sigma^2): -\infty < \mu < \infty, \sigma^2 > 0\}$

$\Omega_0 = \{(\mu, \sigma^2): -\infty < \mu < \infty, 0 < \sigma^2 \le \sigma_0^2\}$

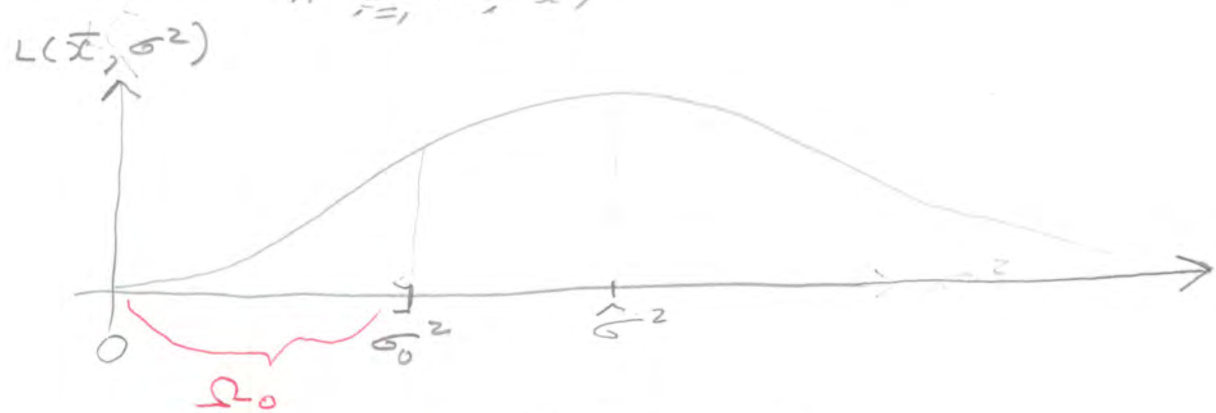
$\Omega_1 = \{(\mu, \sigma^2): -\infty < \mu < \infty, \sigma^2 > \sigma_0^2 > 0\}$

Critical Region $C = \{\underset{\sim}{x} \in \mathbb{R}^n: \lambda(\underset{\sim}{x}) \le k\}$

$= \left\{\underset{\sim}{x}: \dfrac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})} \le k\right\}$

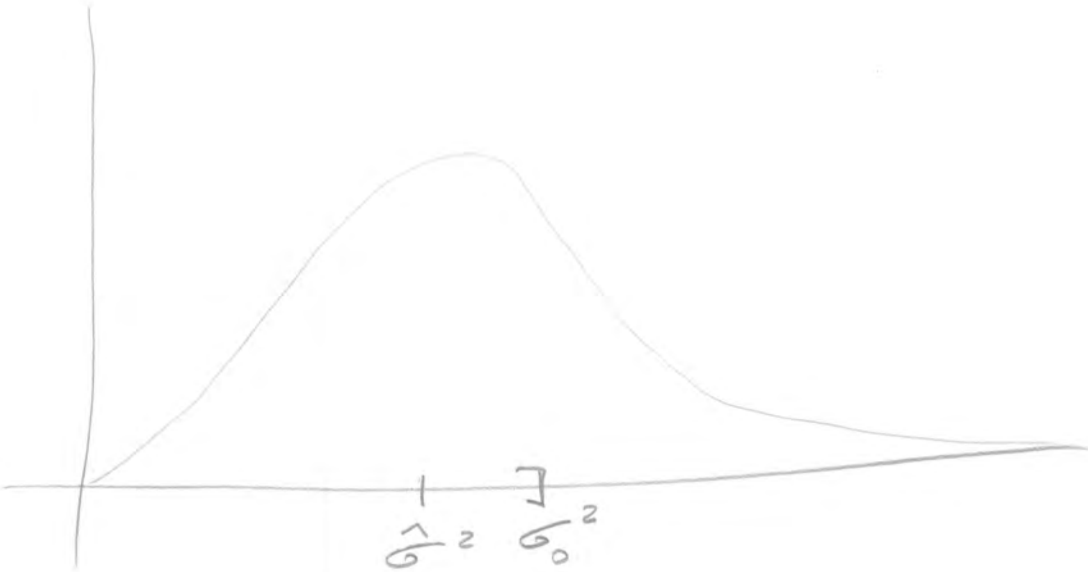
Take advantage of what we already know about this Problem.

1) For any $\sigma^2 > 0$, mle of μ is $\bar{x}$. $\hat{\mu} = \hat{\mu}_0 = \bar{x}$
2) For $\mu = \bar{x}$, likelihood has a unique maximum at $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$



So $\hat{\sigma}_0^2 = \sigma_0^2$

The other possibility is $\hat{\sigma}^2 \le \sigma_0^2$



In that case $\hat{\sigma}_0^2 = \hat{\sigma}^2$,

$\hat{\Theta}_0 = (\bar{X}, \hat{\sigma}^2) = \hat{\Theta}$,

$C = \left\{ \underset{\sim}{x} : \dfrac{L(\hat{\Theta}_0)}{L(\hat{\Theta})} \le k \right\} = \{ x : 1 \le k \}$

where $0 < k < 1$

$= \emptyset$, and H_0 is not rejected

Because $\underset{\sim}{X}$ cannot fall into $\emptyset$

$$\hat{\Theta} = (\bar{x}, \hat{\sigma}^2), \quad \hat{\theta}_0 = (\bar{x}, \sigma_0^2)$$ 13

$$\lambda(\underset{\sim}{x}) = \frac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})} = \frac{\prod_{i=1}^{n} \frac{1}{\hat{\sigma}_0 \cancel{\sqrt{2\pi}}} e^{-\frac{1}{2\hat{\sigma}_0^2}(x_i - \hat{\mu}_0)^2}}{\prod_{i=1}^{n} \frac{1}{\hat{\sigma} \cancel{\sqrt{2\pi}}} e^{-\frac{1}{2\hat{\sigma}^2}(x_i - \hat{\mu})^2}}$$

$$= \left(\frac{\hat{\sigma}}{\sigma_0}\right)^n \frac{e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i - \bar{x})^2}}{e^{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \frac{e^{-\frac{1}{2\sigma_0^2} n\hat{\sigma}^2}}{e^{-\frac{1}{2\cancel{\hat{\sigma}^2}} n\cancel{\hat{\sigma}^2}}}$$

$$= \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \frac{e^{-\frac{n\hat{\sigma}^2}{2\sigma_0^2}}}{e^{-n/2}}$$

$$C = \{\underset{\sim}{x} : \lambda(\underset{\sim}{x}) \le k\}$$

$$= \left\{\underset{\sim}{x} : \left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)^{n/2} \frac{e^{-\frac{n\hat{\sigma}^2}{2\sigma_0^2}}}{e^{-n/2}} \le k\right\}$$

$$= \left\{ \underset{\sim}{x} : \left(\frac{\hat{\sigma}^2}{\sigma_0^2} \right)^{n/2} \left(e^{-\frac{\hat{\sigma}^2}{\sigma_0^2}} \right)^{n/2} \le e^{-n/2} k = k_1 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{\hat{\sigma}^2}{\sigma_0^2} e^{-\frac{\hat{\sigma}^2}{\sigma_0^2}} \le (k_1)^{\frac{2}{n}} = k_2 \right\}$$

Wish we could isolate $\frac{\hat{\sigma}^2}{\sigma_0^2}$.

Consider the function $g(x) = x e^{-x}$, $x \ge 0$

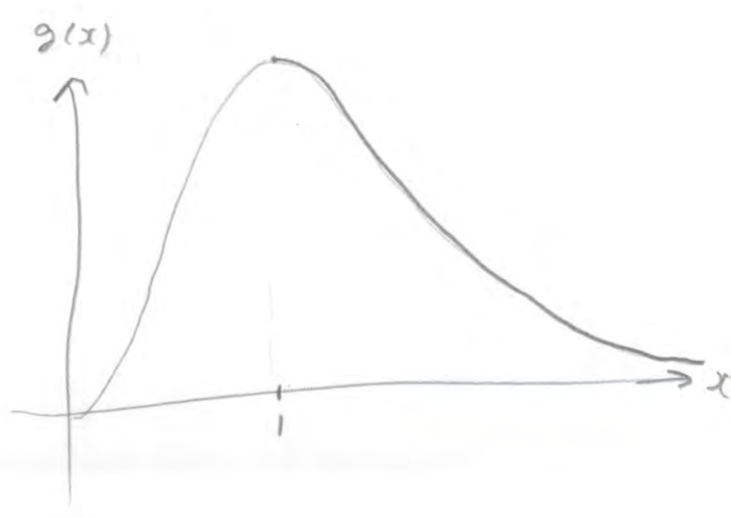
$$g'(x) = u'v + v'u = e^{-x} + x e^{-x} \cdot (-1)$$

$$= e^{-x}(1 - x) \quad = 0 \text{ when } x = 1,$$

and for $x > 1$ it's decreasing. Since $\hat{\sigma}^2 > \sigma_0^2$,

$g(x) = x e^{-x}$ is decreasing at $x = \frac{\hat{\sigma}^2}{\sigma_0^2}$

A strictly decreasing function has an inverse that is also strictly decreasing.



This page was added in response to student questions.

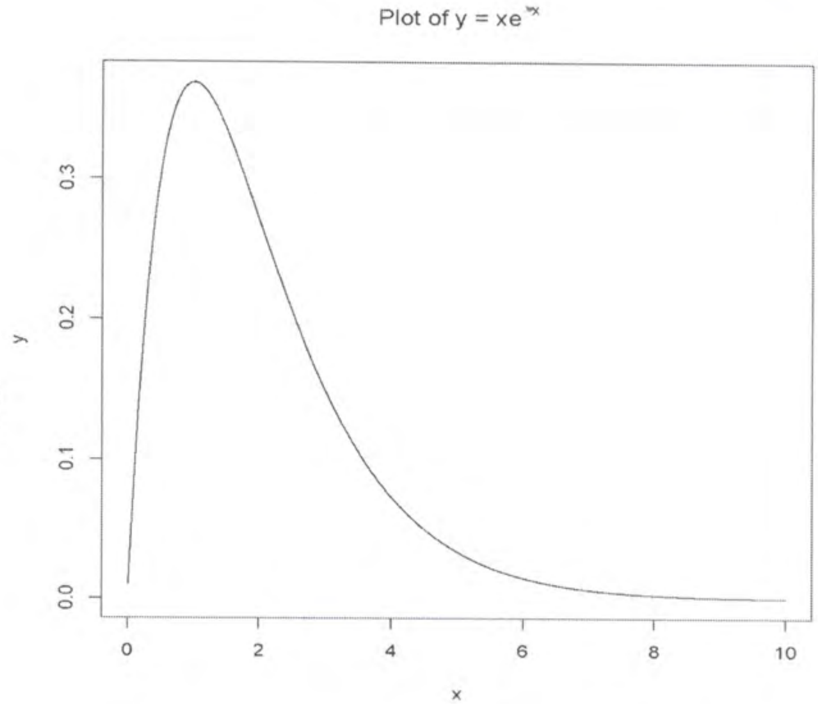
I want to say

$$C = \{ \underset{\sim}{x} : \frac{\hat{\sigma}^2}{\sigma_0^2} e^{-\frac{\hat{\sigma}^2}{\sigma_0^2}} \le k_2 \}$$

$$= \{ \underset{\sim}{x} : g(\frac{\hat{\sigma}^2}{\sigma_0^2}) \le k_2 \}$$

$$= \{ \underset{\sim}{x} : g^{-1}(g(\frac{\hat{\sigma}^2}{\sigma_0^2})) \ge g^{-1}(k_2) = k_3 \}$$

$$= \{ \underset{\sim}{x} : \frac{\hat{\sigma}^2}{\hat{\sigma}_0} \ge k_3 \}$$



But is it valid ?

From the picture, $g^{-1}(y)$ takes a point on the y axis and returns an $x \ge 1$. But g^{-1} is defined only for $y \le 1 \cdot e^{-1} = e^{-1}$. We want to take g^{-1} on both sides of $g(\frac{\hat{\sigma}^2}{\sigma_0^2}) \le k_2$. The left side is no trouble, but do we know that $g^{-1}(k_2)$ is defined? Need $k_2 \le e^{-1}$.

Trace back what k_2 is. Top of p. 14,

$$0 < k < 1$$

$$\Rightarrow 0 < \underbrace{e^{-n/2} k}_{k_1} < e^{-n/2}$$

$$\Rightarrow 0 < \underbrace{k_1^{2/n}}_{k_2} < (e^{-n/2})^{2/n}$$

$$\Rightarrow 0 < k_2 < e^{-1}$$ OKAY ! So, ...

$$C = \left\{ \underset{\sim}{x} : \frac{\hat{\sigma}^2}{\sigma_0^2} e^{-\hat{\sigma}^2/\sigma_0^2} \le k_2 \right\}$$

$$= \left\{ \underset{\sim}{x} : g\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right) \le k_2 \right\}$$

$$= \left\{ \underset{\sim}{x} : g^{-1}\left(g\left(\frac{\hat{\sigma}^2}{\sigma_0^2}\right)\right) \ge g^{-1}(k_2) = k_3 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{\hat{\sigma}^2}{\sigma_0^2} \ge k_3 \right\}$$

$$= \left\{ x : \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2} \ge k_3 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{n\,\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sigma_0^2} \ge n k_3 = k_4 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{(n-1)s^2}{\sigma_0^2} \ge k_4 \right\}$$

If $\sigma^2 = \sigma_0^2$, letting $k_4 = \chi^2_{1-\alpha}(n-1)$ will give significance level α.

But significance level is $\max_{\theta \in \Omega_0} P_\theta(X \in C)$

Must establish $\max_{\sigma^2 \le \sigma_0^2} P_{\sigma^2}(Y \ge k_4) = P_{\sigma_0^2}(Y \ge k_4)$

Suppose true variance is $\sigma^2 \le \sigma_0^2$

$$P_{\sigma^2}\left(\frac{(n-1)S^2}{\sigma_0^2} \ge c\right) =$$

$$= P_{\sigma^2}\left(\frac{(n-1)S^2}{\sigma_0^2} \cdot \frac{\sigma_0^2}{\sigma^2} \ge c\frac{\sigma_0^2}{\sigma^2}\right)$$

$$= P_{\sigma^2}\left(\frac{(n-1)S^2}{\sigma^2} \ge c\frac{\sigma_0^2}{\sigma^2}\right)$$

$$= 1 - F_W\left(c\frac{\sigma_0^2}{\sigma^2}\right) \text{ where } W \sim \chi^2(n-1)$$

A function of σ^2 Increasing ?

σ_0^2

$$\frac{d}{d\sigma^2}\left(1 - F_W\left(c\sigma_0^2(\sigma^2)^{-1}\right)\right)$$

$$= -f_W\left(\frac{c\sigma_0^2}{\sigma^2}\right) * c\sigma_0^2(-1)(\sigma^2)^{-2}$$

$$= f_W\left(\frac{c\sigma_0^2}{\sigma^2}\right)\frac{c\sigma_0^2}{\sigma^4} > 0 \quad \uparrow$$

And $\max_{\theta \in \Omega_0}$ occurs at $\sigma^2 = \sigma_0^2$

So to make test size α,

$$C = \left\{ \underset{\sim}{x} \in \mathbb{R}^n : \frac{(n-1)s^2}{\sigma_0^2} \geq \chi^2_{1-\alpha}(n-1) \right\}$$

Test statistic for exact likelihood ratio test is

$$W = \frac{(n-1)S^2}{\sigma_0^2}$$

Reject H_0 & conclude $\sigma^2 > \sigma_0^2$ when

$$W \geq \chi^2_{1-\alpha}(n-1)$$

<u>Comment</u> Exact likelihood ratio tests are a lot of work to prove, but they are very high quality - high power.