Tuesday March 17th

Likelihood Ratio Tests: A general principle for deriving tests.

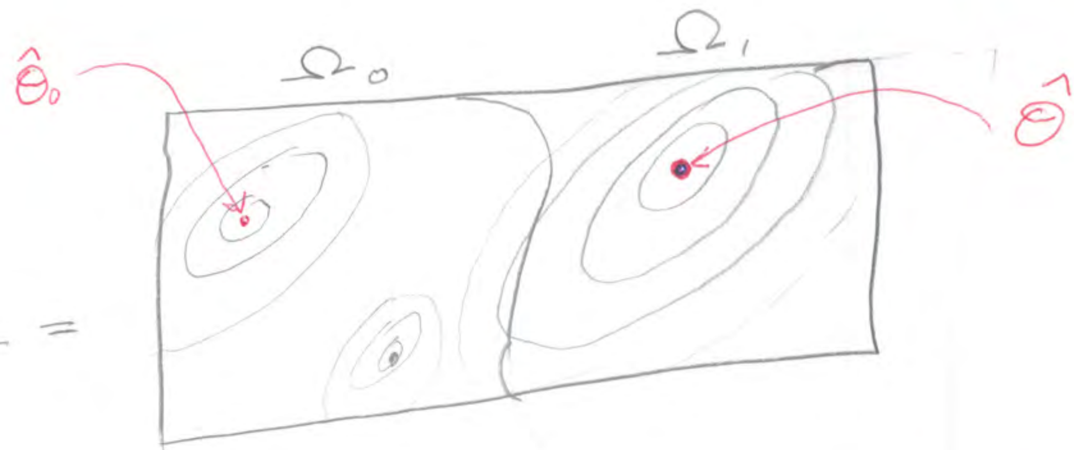
$\underset{\sim}{X} \sim P_\theta$, $\theta \in \Omega$, $H_0: \theta \in \Omega_0$ vs. $H_1: \theta \in \Omega_1$,

$\Omega = \Omega_0 \cup \Omega_1$, disjoint

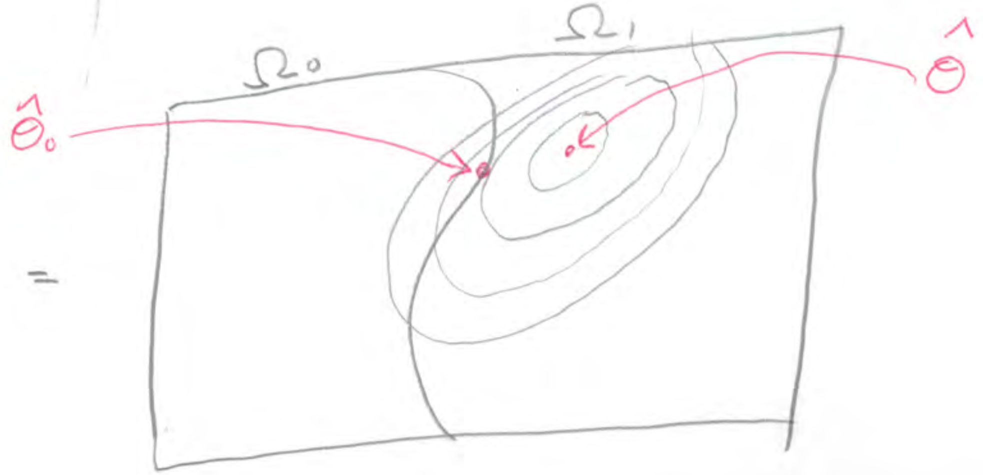
Find TWO MAXIMUM LIKELIHOOD ESTIMATES

- One over all of Ω: get $\hat{\theta}$
- One over just Ω_0, get $\hat{\theta}_0$

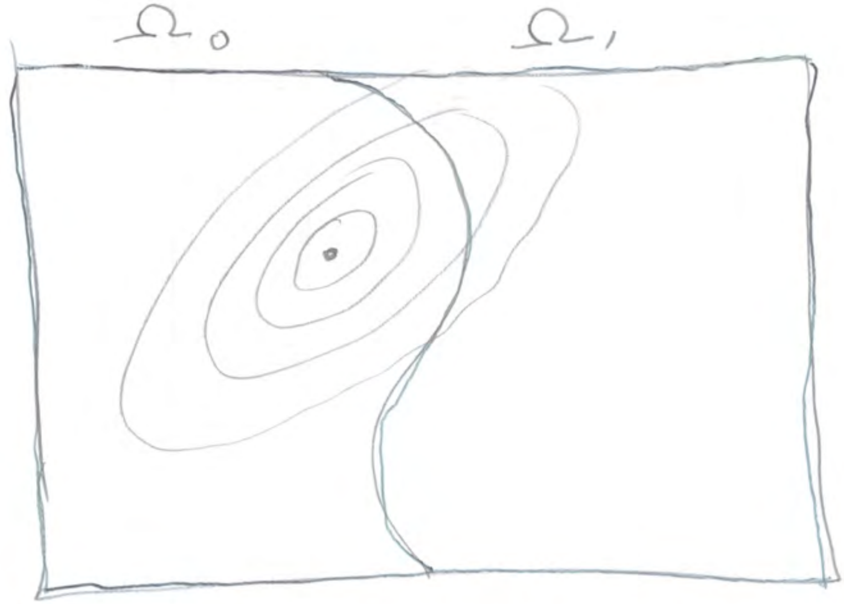
$\Omega =$



$\Omega =$



If the overall MLE is in Ω_0,
$\hat{\theta}_0 = \hat{\theta}$



And you would not want to reject H_0

Critical Region

$$C = \left\{ \underset{\sim}{x} \in S : \frac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})} \leq k \right\} \quad \text{where } 0 < k < 1$$

$$= \left\{ \underset{\sim}{x} \in S : \lambda(\underset{\sim}{x}) \leq k \right\}$$

- Because $L(\hat{\theta}_0, \underset{\sim}{x}) \leq L(\hat{\theta}, x)$, $\lambda(\underset{\sim}{x}) \leq 1$

- How much less? If a lot less, the observed data are SURPRISING if H_0 is true, and that's evidence against H_0

- Choose k so test has significance level α. That is, so $\max_{\theta \in \Omega_0} P_\theta(X \in C) = \alpha$

$$C = \left\{ \underset{\sim}{x} \in S : \frac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})} \leq k \right\}$$

Two approaches to making the test size α

EXACT : Play with C until you can express it in terms of a statistic whos distribution you know under H_0

LARGE-SAMPLE (In case of failure to find exact)

$$\lambda(x) = \frac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})}, \quad \Lambda_n(\underset{\sim}{X}) = \frac{L(\hat{\Theta}_0, \underset{\sim}{X})}{L(\hat{\Theta}, \underset{\sim}{X})}$$

Under H_0, $-2 \ln \Lambda_n(\underset{\sim}{X}) \xrightarrow{d} Y \sim \chi^2(\quad)$

So a large-sample likelihood ratio test will be based on

$$C = \left\{ \underset{\sim}{x} : -2 \ln \frac{L(\hat{\theta}_0, \underset{\sim}{x})}{L(\hat{\theta}, \underset{\sim}{x})} \geq \chi^2_{1-\alpha}(\quad) \right\}$$

More about the df later.

Exact likelihood ratio tests are better if available.

<u>Example:</u> $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known

$H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$

$\Omega = \mathbb{R}$, $\Omega_0 = \{\mu_0\}$, $\Omega_1 = (-\infty, \mu_0) \cup (\mu_0, \infty)$

$\hat{\mu} = \bar{x}$, $\hat{\mu}_0 = \mu_0$

$$C = \left\{ \underset{\sim}{x} \in \mathbb{R}^n : \frac{L(\hat{\mu}_0, \underset{\sim}{x})}{L(\hat{\mu}, \underset{\sim}{x})} \le k \right\}, \quad 0 < k < 1$$

$$= \left\{ \underset{\sim}{x} : \frac{\prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu_0)^2}}{\prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\bar{x})^2}} \le k \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{\cancel{\frac{1}{\sigma^n(2\pi)^{n/2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}}{\cancel{\frac{1}{\sigma^n(2\pi)^{n/2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{x})^2}} \le k \right\}$$

$$= \left\{ \underset{\sim}{x} : \ln\left(\frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{x})^2}} \right) \le \ln k = k_1 \right\}$$

$$= \left\{ \underset{\sim}{x} : -\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2 - -\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\bar{x})^2 \le k_1 \right\}$$

Multiply both sides by $-2\sigma^2$

$$= \{ \underset{\sim}{x} : \sum_{i=1}^{n} (x_i - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \bar{x})^2 \geq -2\sigma^2 k_1 = k_2 \}$$

$$= \{ \underset{\sim}{x} : \sum_{i=1}^{n} (x_i - \bar{x} + \bar{x} - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \bar{x})^2 \geq k_2 \}$$

$$= \{ \underset{\sim}{x} : \cancel{\sum_{i=1}^{n} (x_i - \bar{x})^2} + n(\bar{x} - \mu_0)^2 - \cancel{\sum_{i=1}^{n} (x_i - \bar{x})^2} \geq k_2 \}$$

$$= \{ \underset{\sim}{x} : \frac{n(\bar{x} - \mu_0)^2}{\sigma^2} \geq \frac{k_2}{\sigma^2} = k_3 \}$$

$$= \{ \underset{\sim}{x} : \left(\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right)^2 \geq k_3 \}$$

Now can pick k_3 so that if $H_0 : \mu = \mu_0$ is true

$$P(\underset{\sim}{X} \in C) = P\left(\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \geq k_3 \right) = \alpha$$

Let $k_3 = \chi^2_{1-\alpha}(1)$ Chi-squared test

OR

$$C = \{ \underset{\sim}{x} : \left(\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right)^2 \geq k_3 \}$$

$$= \{ \underset{\sim}{x} : \sqrt{\left(\frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right)^2} \geq \sqrt{k_3} = k_4 \}$$

$$= \{ \underset{\sim}{x} : \left| \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \right| \geq k_4 \}$$ & let $k_4 = z_{1-\alpha/2}$

2-sided Z-test

Example Independently for $j=1,\dots,k$ and $i=1,\dots,n_j$

$$X_{ij} \sim N(\mu_j, \sigma^2) \qquad H_0: \mu_1 = \mu_2 = \cdots = \mu_k$$

Need $\hat\theta$ & $\hat\theta_0$. Try to avoid work we have already done. Know if $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$,
$\hat\mu = \bar x$ & $\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar x)^2$

If H_0 is true, $\mu_1 = \mu_2 = \cdots = \mu_k = \mu$, and the X_{ij} are one big random sample:

$$\hat\mu_0 = \bar X_{\cdot} = \sum_{j=1}^k \frac{n_j}{n}\bar X_j = \frac{1}{n}\sum_{j=1}^k\sum_{i=1}^{n_j} X_{ij} \quad \text{And}$$

$$\hat\sigma_0^2 = \frac{1}{n}\sum_{j=1}^k\sum_{i=1}^{n_j}(X_{ij} - \bar X_{\cdot})^2 = \frac{1}{n}\,SSTO$$

If H_0 is false, μ_j could be different.

$$\frac{\partial}{\partial\mu_1}\ln L(\mu, \sigma^2, \underset{\sim}{x}) = \frac{\partial}{\partial\mu_1}\ln\prod_{j=1}^k\prod_{i=1}^{n_j}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(x_{ij}-\mu_j)^2}$$

$$= \frac{\partial}{\partial\mu_1}\sum_{j=1}^k \ln\prod_{i=1}^{n_j} e^{-\frac{1}{2\sigma^2}(x_{ij}-\mu_j)^2}$$

$$= \sum_{j=1}^k \frac{\partial}{\partial\mu_1}\ln\prod_{i=1}^{n_j} e^{-\frac{1}{2\sigma^2}(x_{ij}-\mu_j)^2}$$

$$= \frac{\partial}{\partial\mu_1}\ln\prod_{i=1}^{n_1} e^{-\frac{1}{2\sigma^2}(x_{i1}-\mu_1)^2} + 0 + \cdots + 0$$

So $\hat{\mu}_1 = \bar{x}_1$, $\hat{\mu}_2 = \bar{x}_2$, --, $\hat{\mu}_k = \bar{x}_k$

For any σ^2

$$\frac{\partial}{\partial\sigma} \ell(\underset{\sim}{\bar{x}}, \sigma^2, \underset{\sim}{x}) = \frac{\partial}{\partial\sigma} \ln \prod_{j=1}^{k} \prod_{i=1}^{n_j} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_{ij}-\bar{x}_j)^2}$$

$$= \frac{\partial}{\partial\sigma} \ln \left(\sigma^{-n} (2\pi)^{-n/2} e^{-\frac{1}{2\sigma^2} \sum_{j=1}^{k}\sum_{i=1}^{n_j} (x_{ij}-\bar{x}_j)^2} \right)$$

$$= \frac{\partial}{\partial\sigma} \left(-n \ln\sigma - \frac{n}{2} \ln 2\pi - \frac{1}{2} SSW \cdot \sigma^{-2} \right)$$

$$= \frac{-n}{\sigma} - 0 - \frac{1}{2} SSW (-2) \sigma^{-3}$$

$$= \frac{-n}{\sigma} + \frac{SSW}{\sigma^3} \overset{set}{=} 0 \Rightarrow \frac{n}{\sigma} = \frac{SSW}{\sigma^3}$$

$$\Rightarrow \sigma^2 = \frac{SSW}{n}$$

2nd derivative test

$$\frac{\partial}{\partial\sigma} \left(-n\sigma^{-1} + SSW\sigma^{-3} \right)$$

$$= (-n)(-1)\sigma^{-2} - 3 SSW \sigma^{-4}$$

$$= \frac{n}{\sigma^2} - \frac{3 \cdot SSW}{\sigma^4}$$ Evaluating at $\sigma^2 = \frac{1}{n} SSW$

$$= \frac{n}{SSW/n} - \frac{3 SSW}{(\frac{1}{n} SSW)^2} = \frac{n^2}{SSW} - \frac{3n^2}{SSW} < 0$$ CCD MAX

So $\hat{\Theta} = (\hat{\mu}_1, \ldots, \hat{\mu}_k, \hat{\sigma}^2) = (\bar{x}_1, \ldots, \bar{x}_k, \frac{1}{n} SSW)$

And $\hat{\Theta}_0 = (\bar{x}_{\cdot}, \ldots, \bar{x}_{\cdot}, \frac{1}{n} SSTO)$

Notice Θ, $\hat{\Theta}$ & $\hat{\Theta}_0$ are all length $k+1$

Likelihood Ratio $\quad \lambda(\underset{\sim}{x}) = \dfrac{L(\hat{\Theta}_0, \underset{\sim}{x})}{L(\hat{\Theta}, \underset{\sim}{x})}$

$$\lambda(\underset{\sim}{x}) = \frac{\prod_{j=1}^{k}\prod_{i=1}^{n_j} \frac{1}{\hat{\sigma}_0 \cancel{\sqrt{2\pi}}} e^{-\frac{1}{2\hat{\sigma}_0^2}(x_{ij}-\bar{x}_{\cdot})^2}}{\prod_{j=1}^{k}\prod_{i=1}^{n_j} \frac{1}{\hat{\sigma} \cancel{\sqrt{2\pi}}} e^{-\frac{1}{2\hat{\sigma}^2}(x_{ij}-\bar{x}_j)^2}}$$

$$= \frac{\hat{\sigma}^n}{\hat{\sigma}_0^n} \frac{e^{-\frac{1}{2}\frac{1}{SSTO/n}\overbrace{\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_{\cdot})^2}^{SSTO}}}{e^{-\frac{1}{2}\frac{1}{SSW/n}\underbrace{\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij}-\bar{x}_j)^2}_{SSW}}}$$

$$= \frac{\hat{\sigma}^n}{\hat{\sigma}_0^n} \frac{e^{-n/2}}{e^{-n/2}} = \left(\frac{SSW/n}{SSTO/n}\right)^n$$

$$C = \left\{ \underset{\sim}{x} : \left(\frac{SSW}{SSTO}\right)^n \le k \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{SSW}{SSTO} \le k^{1/n} = k_1 \right\}$$

$$C = \left\{ \underset{\sim}{x} : \frac{SSW}{SSTO} \le k_1 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{SSTO - SSB}{SSTO} \le k_1 \right\}$$

Recall $R^2 = \dfrac{SSB}{SSTO}$ & $F = \left(\dfrac{n-k}{k-1}\right)\left(\dfrac{R^2}{1-R^2}\right)$

$$= \left\{ \underset{\sim}{x} : 1 - R^2 \le k_1 \right\}$$

Consider the function $g(x) = \dfrac{1-x}{x} = \dfrac{1}{x} - 1$

Decreasing in x for $x > 0$

$g(1-R^2) = \dfrac{1-(1-R^2)}{1-R^2} = \dfrac{R^2}{1-R^2}$, so

$$C = \left\{ \underset{\sim}{x} : 1 - R^2 \le k_1 \right\} = \left\{ \underset{\sim}{x} : g(1-R^2) \ge g(k_1) = k_2 \right\}$$

$$= \left\{ \underset{\sim}{x} : \frac{R^2}{1-R^2} \ge k_2 \right\}$$

$$= \left\{ \underset{\sim}{x} : \left(\frac{n-k}{k-1}\right)\left(\frac{R^2}{1-R^2}\right) \ge \left(\frac{n-k}{k-1}\right) k_2 = k_3 \right\}$$

$$= \left\{ \underset{\sim}{x} : F \ge k_3 \right\}$$ Let $k_3 = f_{1-\alpha}(k-1, n-k)$

THE F-TEST FOR k INDEPENDENT MEANS IS AN EXACT LIKELIHOOD RATIO TEST