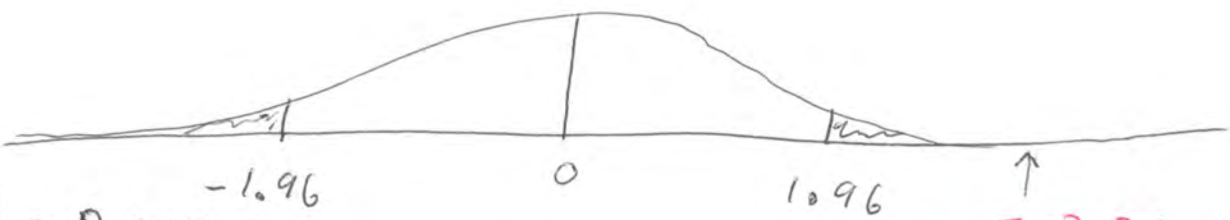
Tuesday March 3

Had $Y_i = \beta x_i + E_i$, $E_1, \dots, E_n \overset{iid}{\sim} N(0, \sigma^2)$

Known

$H_0: \beta = 0$ vs $H_1: \beta \neq 0$, $\alpha = 0.05$

Two-sided alternative, 2-tailed test



$\hat{\beta} = 0.429$

Found $Z = 3.29$, Rejected H_0. What do we conclude? $\beta \neq 0$, or $\beta > 0$?

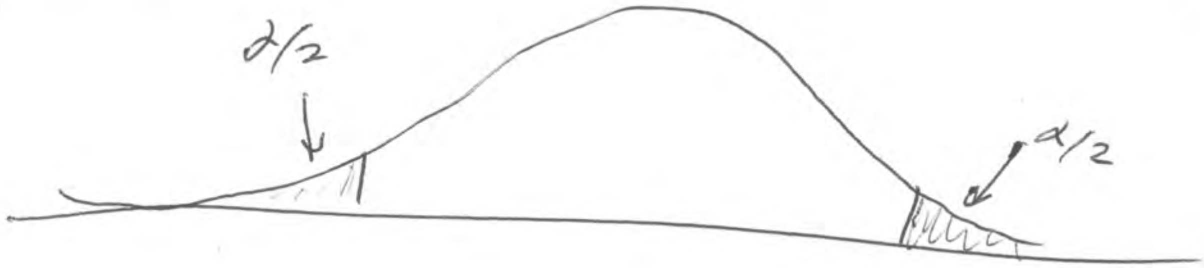
Ex 2 sleeping drugs Take drug wait Take the other drug

Drug A Sleep	Drug B Sleep
6.2	5.1
8.0	9.7
⋮	

Compute $X_1 \dots X_n$ differences

$H_0: \mu = 0$

Solution:



Decompose into

A) $H_0: \mu \leq \mu_0$ vs $H_1: \mu > \mu_0$

B) $H_0: \mu \geq \mu_0$ vs H_1 $\mu < \mu_0$

Sig level: $\frac{\alpha}{2}$

H_0 will be rejected with 2-sided test iff one of the 1-sided tests Rejects H_0, & can draw conclusion from one-sided test.

Ass 4

29

4. Again, let $X_1, \ldots, X_n$ be random sample from a $N(\mu, \sigma^2)$ distribution.

   (a) The t distribution is defined as follows. Let $Z \sim N(0,1)$ and $Y \sim \chi^2(\nu)$, with Z and Y independent. Then $T = \frac{Z}{\sqrt{Y/\nu}}$ is said to have a t distribution with ν degrees of freedom, and we write $T \sim t(\nu)$. Using results from earlier questions, prove $T = \frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t(n-1)$. Be sure to indicate why your Z and Y are independent.

   (b) You can see that T is a "pivotal quantity." It's a random variable that is a function of the parameter, but whose distribution does not depend on the parameter value. Starting with a probability statement about the pivotal quantity, derive an exact $(1-\alpha)100\%$ confidence interval for μ. "Derive" means show all the high school algebra. Your answer is a pair of formulas, one for the lower confidence limit and one for the upper confidence limit.

   (c) The t distribution was introduced by William Gossett, writing under the name Student. The reference is Student (1908). "The probable error of a mean," *Biometrika* 6, 1-25. Gossett illustrated the method using two measurements on ten patients suffering from insomnia (trouble sleeping). Each number is a difference, representing how much *extra* sleep the patient got when taking a sleeping pill, compared to a baseline measurement. Drug 1 is Dextro-hyoscyamine hydrobomide, while Drug 2 is Laevo-hyoscyamine hydrobomide. Each patent tried both drugs, with a recovery period of several days between trials. Here are the data:

Patient	Drug 1	Drug 2
1	0.7	1.9
2	-1.6	0.8
3	-0.2	1.1
4	-1.2	0.1
5	-0.1	-0.1
6	3.4	4.4
7	3.7	5.5
8	0.8	1.6
9	0.0	4.6
10	2.0	3.4

   The strategy here is to compute a difference for each patient, Drug 1 minus Drug 2. The difference represents how much more sleep the patient got when using Drug 1. The differences are $X_1, \ldots, X_{10}$, and μ is the expected advantage of Drug 1 over Drug 2. Give a point estimate and a 95% confidence interval for μ. The point estimate is a single number. The confidence interval is a pair of numbers, the lower confidence limit and the upper confidence limit.

   (d) Does the confidence interval allow you to decide which drug worked better?

   (e) Derive an exact $(1-\alpha)100\%$ confidence interval for σ^2. "Derive" means show all the high school algebra. Your answer is a pair of formulas, one for the lower confidence limit and one for the upper confidence limit. You can locate the pivotal quantity in your answers to earlier questions.

   (f) Using the data from Question (4c), give a 95% confidence interval for σ^2. The answer is a pair of numbers, the lower confidence limit and the upper confidence limit.

One-sample t-test

Model: $X_1, \ldots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

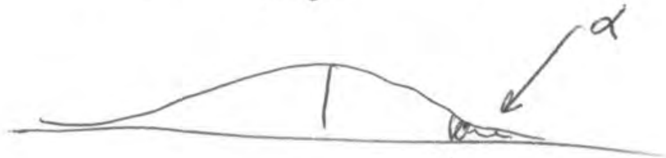
Null hypothesis: $\mu = \mu_0$, $\mu \le \mu_0$, $\mu \ge \mu_0$

Test statistic

$$T = \frac{\sqrt{n}\,(\bar{X}_n - \mu_0)}{S} \overset{H_0}{\sim} t(n-1)$$

Decision Rule

H_0	H_1	Reject when
$\mu = \mu_0$	$\mu \ne \mu_0$	$\lvert T \rvert \ge t_{1-\alpha/2}(n-1)$
$\mu \le \mu_0$	$\mu > \mu_0$	$T \ge t_{1-\alpha}(n-1)$
$\mu \ge \mu_0$	$\mu < \mu_0$	$T \le t_\alpha = -t_{1-\alpha}$



$t_{1-\alpha}$

$t_\alpha = -t_{1-\alpha}$

Most common application:
Matched, or Paired t-test

X_i are differences, computed on same individuals

Drug 1	Drug 2	D	D^2
0.7	1.9	-1.2	1.44
-1.6	0.8	-2.4	5.76
-0.2	1.1	-1.3	1.69
-1.2	0.1	-1.3	1.69
-0.1	-0.1	0.0	0.00
3.4	4.4	-1.0	1.00
3.7	5.5	-1.8	3.24
0.8	1.6	-0.8	0.64
0.0	4.6	-4.6	21.16
2.0	3.4	-1.4	1.96
$\bar{X}$	$\bar{Y}$	-15.8	38.58

$$S^2 = \frac{\sum_{i=1}^{n}(D_i - \bar{D})^2}{n-1} = \frac{1}{n-1}\left(\sum_{i=1}^{n} D_i^2 - n\bar{D}^2\right)$$

$\bar{D} = -1.58$

$$= \frac{1}{9}\left(38.58 - 10 \times (-1.58)^2\right) = 1.513$$

$$t = \frac{\sqrt{n}\,(\bar{X} - \mu_0)}{S} = \frac{\sqrt{10}\,(-1.58 - 0)}{\sqrt{1.513}}$$

$$= -4.06 < -2.262$$

$t_{0.975}^{(9)} = 2.262$

Remember

$$T = \frac{\sqrt{n}\,(\bar{X} - \mu_0)}{S} \sim t(n-1)$$

If $Z \sim N(0,1)$ & $W \sim \chi^2(r)$ ind.

Then $T = \dfrac{Z}{\sqrt{W/r}} \sim t(r)$

$$W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{Why?}$$

$Y = Y_1 + Y_2$, Y_1 & Y_2 ind, $Y \sim \chi^2(r_1 + r_2)$

Then $Y_1 \sim \chi^2(r_1)$ $\qquad Y_2 \sim \chi^2(r_2)$

<u>And</u>

$$\sum_{i=1}^{n}(X_i - \mu)^2 = \sum_{i=1}^{n}(X_i - \bar{X} + \bar{X} - \mu)^2 = \cdots =$$

$$= \sum_{i=1}^{n}(X_i - \bar{X})^2 + n(\bar{X}_n - \mu)^2$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 + \frac{1}{\sigma^2}n(\bar{X}_n - \mu)^2$$

$$\Rightarrow \overbrace{\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2}^{Y} = \overbrace{\frac{(n-1)S^2}{\sigma^2}}^{Y_1} + \overbrace{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2}^{Y_2}$$

$\chi^2(n) \qquad \chi^2(n-1) \qquad \chi^2_1$

Have $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$T = \frac{Z}{\sqrt{W/r}} = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} = \frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{S/\sigma}$$

$$= \frac{\sqrt{n}(\bar{X}-\mu_0)}{S} \quad \text{If } \mu=\mu_0$$

What if data not normal?
n large

Good for $n \geq 25$?

By-product: Test for variance (34)

$H_0: \sigma^2 = \sigma_0^2$ or $\sigma^2 \le \sigma_0^2$ or $\sigma^2 \ge \sigma_0^2$

Test statistic: $Y = \dfrac{(n-1)S^2}{\sigma_0^2} \overset{H_0}{\sim} \chi^2(n-1)$

Ex Quality control

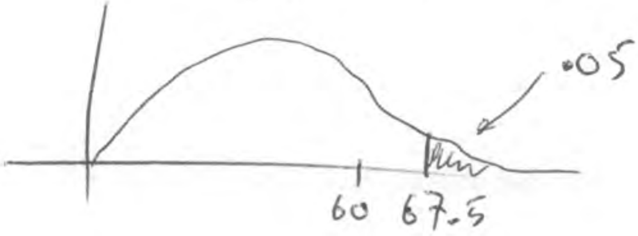
Machine screws are supposed to be 10mm in diameter. Average diameter is almost always ok, But if process is working properly, $\sigma^2 \le 0.01$ mm.

Sample of 51 screws yields $\bar{x} = 9.9994$ mm, $s^2 = 0.012$. Is it OKAY?

$H_0: \sigma^2 \le 0.01$, $H_1: \sigma^2 > 0.01$

$$Y = \frac{(51-1)(0.012)}{.01} = 60 \qquad \alpha = 0.05$$

critical value $\chi^2_{.95}(50) = 67.5$



Accept H_0, Go home

How do you know what test statistic to use?
Try pivotal quantites from confidence intervals.

<u>Theorem</u> (For t-tests & z-tests)

$H_0: \theta = \theta_0$ will be rejected in favour of $H_1: \theta \neq \theta_0$ at significance level α iff θ_0 is <u>outside</u> the $(1-\alpha)100\%$ confidence interval.

<u>Proof for t</u> (Minor adaptation for z)

H_0 is <u>not</u> rejected iff

$$-t_{1-\alpha/2} < T < t_{1-\alpha/2}$$

$$\iff -t_{1-\alpha/2} < \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{S} < t_{1-\alpha/2}$$

$$\iff -t_{1-\alpha/2}\frac{S}{\sqrt{n}} < \bar{X}_n - \mu_0 < t_{1-\alpha/2}\frac{S}{\sqrt{n}}$$

$$\iff -\bar{X}_n - t_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}} < -\mu_0 < -\bar{X}_n + t_{1-\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\iff \bar{X}_n + t_{1-\alpha/2}\frac{S}{\sqrt{n}} > \mu_0 > \bar{X}_n - t_{1-\alpha/2}\frac{S}{\sqrt{n}}$$

upper confidence limit — Lower confidence limit

In HW4 $\left.\begin{array}{l} X_1 \cdots X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2) \\ Y_1, \cdots Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2) \end{array}\right\} ind$

Monkey example

Interested in testing $H_0: \mu_1 = \mu_2$ vs $\mu_1 \neq \mu_2$

Less common in practice $\left\{ \begin{array}{l} \text{or } H_0: \mu_1 \le \mu_2 \text{ vs } H_1: \mu_1 > \mu_2 \\ H_0: \mu_1 \ge \mu_2 \text{ vs } H_1: \mu_1 < \mu_2 \end{array} \right.$

Test statistic for INDEPENDENT GROUPS T-TEST

$$T = \frac{\bar{X}_{n_1} - \bar{Y}_{n_2} - d_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{H_0}{\sim} t(n_1 + n_2 - 2)$$

where $$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Proof of distribution: know

$$\bar{X} \sim N(\mu_1, \tfrac{\sigma^2}{n_1})$$
$$\bar{Y} \sim N(\mu_2, \tfrac{\sigma^2}{n_2})$$
$$\frac{(n_1 - 1)S_1^2}{\sigma^2} \sim \chi^2(n_1 - 1), \quad \frac{(n_2 - 1)S_2^2}{\sigma^2} \sim \chi^2(n_2 - 1)$$

All independent

If $Z \sim N(0,1)$, $W \sim \chi^2(\nu)$ ind.

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t(\nu)$$

$$W = \frac{(n_1-1)S_1^2}{\sigma^2} + \frac{(n_2-1)S_2^2}{\sigma^2} \overset{\text{ind}}{\sim} \chi^2(n_1+n_2-2)$$

$$Z = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} \sim N(0,1)$$

d_0: $\mu_1 - \mu_2$ under H_0

$$T = \frac{Z}{\sqrt{W/\nu}} = \frac{\dfrac{\bar{X} - \bar{Y} - d_0}{\cancel{\sigma}\sqrt{1/n_1 + 1/n_2}}}{\dfrac{1}{\cancel{\sigma}}\sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}}}$$

(Behrens-Fisher problem: $\sigma_1^2 \neq \sigma_2^2$)

$$= \frac{\bar{X} - \bar{Y} - d_0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{H_0}{\sim} t(n_1+n_2-2)$$

Monkeys

	n	Mean	SD
Exper Group	11	62.27	10.57
control Group	7	78.57	10.29

$S_1^2 = 10.57^2 = 111.72$

$S_2^2 = 10.29^2 \quad 105.88$

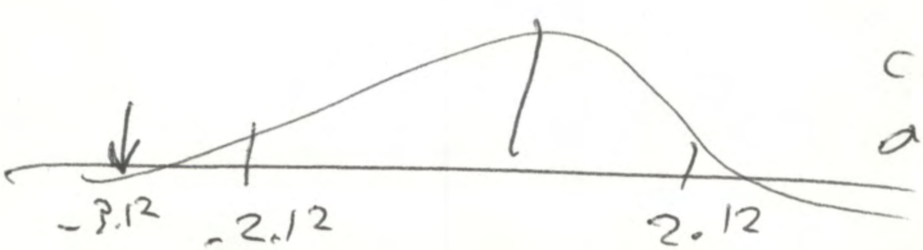
$$S_p^2 = \frac{(11-1)111.72 + (7-1)105.88}{11+7-2} = 109.53$$

$$T = \frac{62.27 - 78.57 - 0}{\sqrt{109.53}\sqrt{\frac{1}{7}+\frac{1}{11}}}$$

$$= \frac{-16.3}{5.06} = -3.22$$

< -2.12

crit value at $\alpha = 0.05$ with $df = 16$



Test for = Variances (39)

$S_1^2 = 105.88 \qquad n_1 = 11$

$S_2^2 = 111.72 \qquad n_2 = 7$

If $W_1 \sim \chi^2(\nu_1)$, $W_2 \sim \chi^2(\nu_2)$
ind

$$F = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F(\nu_1, \nu_2)$$

$$= \cancel{\frac{105.88/10}{111.72/6}} = \frac{111.72}{105.88} = 1.06$$

$< F_{0.975}(10,6)$
" 5.46

$$\frac{\frac{\cancel{(n_1-1)} S_1^2}{\cancel{(n_1-1)}}}{\frac{\cancel{(n_2-1)} S_2^2}{\cancel{(n_2-1)}}} = \frac{S_1^2}{S_2^2}$$