# Thursday Feb. 27th



Surgery wait times $X_1 \dots X_n \overset{iid}{\sim}$ Exponential $(\lambda)$

$H_0: \lambda \geq \frac{1}{2}$ vs $H_1: \lambda < \frac{1}{2} \iff H_0: \mu \leq 2$ vs $H_1: \mu > 2$

$\bar{X}_n \sim Gamma(n, \lambda n)$

Two test statistics

$$Y = 2n\lambda_0 \bar{X}_n \sim \chi^2(2n) \text{ when } \lambda = \lambda_0$$

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\hat{\sigma}_n} \xrightarrow{d} Z \sim N(0,1) \text{ when } \mu = \mu_0$$

With $n = 29$, $\lambda_0 = \frac{1}{2} \iff \mu_0 = 2$, exponential model correct, found that if $\mu = \mu_0 = 2$

$$P(Z_n \geq 1.645) = 0.059$$

Advantages & Disadvantages

$Y$: $\chi^2$ is exact if model is true

$Z$ does not have sig level $\alpha$, but close

If model is wrong — NOT EXPONENTIAL

$Z$ is good for large $n$, $Y$ might be very bad Robustness

If the model is correct, is Y test really better? What do we mean by better?

Higher probability of correctly rejecting null hypothesis. Detect real trend, finding

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**Def** Power of a test is probability of rejecting $H_0$ when $H_0$ is false. It's a function of $\theta$

$P_\theta(\underset{\sim}{X} \in C)$ (when $\theta \in \Omega_1$)

Power $= 1 - P(\text{Type II error})$

For Y test

$$\text{Power}(\lambda) = P_\lambda(Y > \chi^2_{1-\alpha}(2n))$$

$$= P_\lambda(2n\lambda_0 \bar{X}_n \geq \chi^2_{1-\alpha}(2n))$$

$$= P_\lambda(2 * 29 * \tfrac{1}{2} * \bar{X}_n \geq 76.78)$$

$$= P_\lambda(\bar{X}_n \geq 2.65)$$

$\bar{X}_n \sim G(n, n\lambda)$

For $z$ test, if Exponential model is true

Suppose $S^2 \overset{\text{exactly}}{=} \operatorname{var}(X_i) = \frac{1}{\lambda^2}$

$S = \frac{1}{\lambda}$

$\mu = \frac{1}{\lambda}$

$$P_\mu(Z_n \geq 1.645) = P_\mu\left(\frac{\sqrt{n}\,(\bar{X}_n - \mu_0)}{S_n} \geq 1.645\right)$$

suppose

$$\overset{\downarrow}{=} P_\mu\left(\frac{\sqrt{29}\,(\bar{X}_n - 2)}{\frac{1}{\lambda}} \geq 1.645\right)$$

$$= P_\lambda\left(\bar{X}_n \geq 2 + \frac{1}{\lambda\sqrt{29}}\,1.645\right)$$

$$\bar{X}_n \sim G(n, 2\lambda)$$

Compare, as $\lambda$ ranges over $\Omega$, $= (0, \frac{1}{2})$

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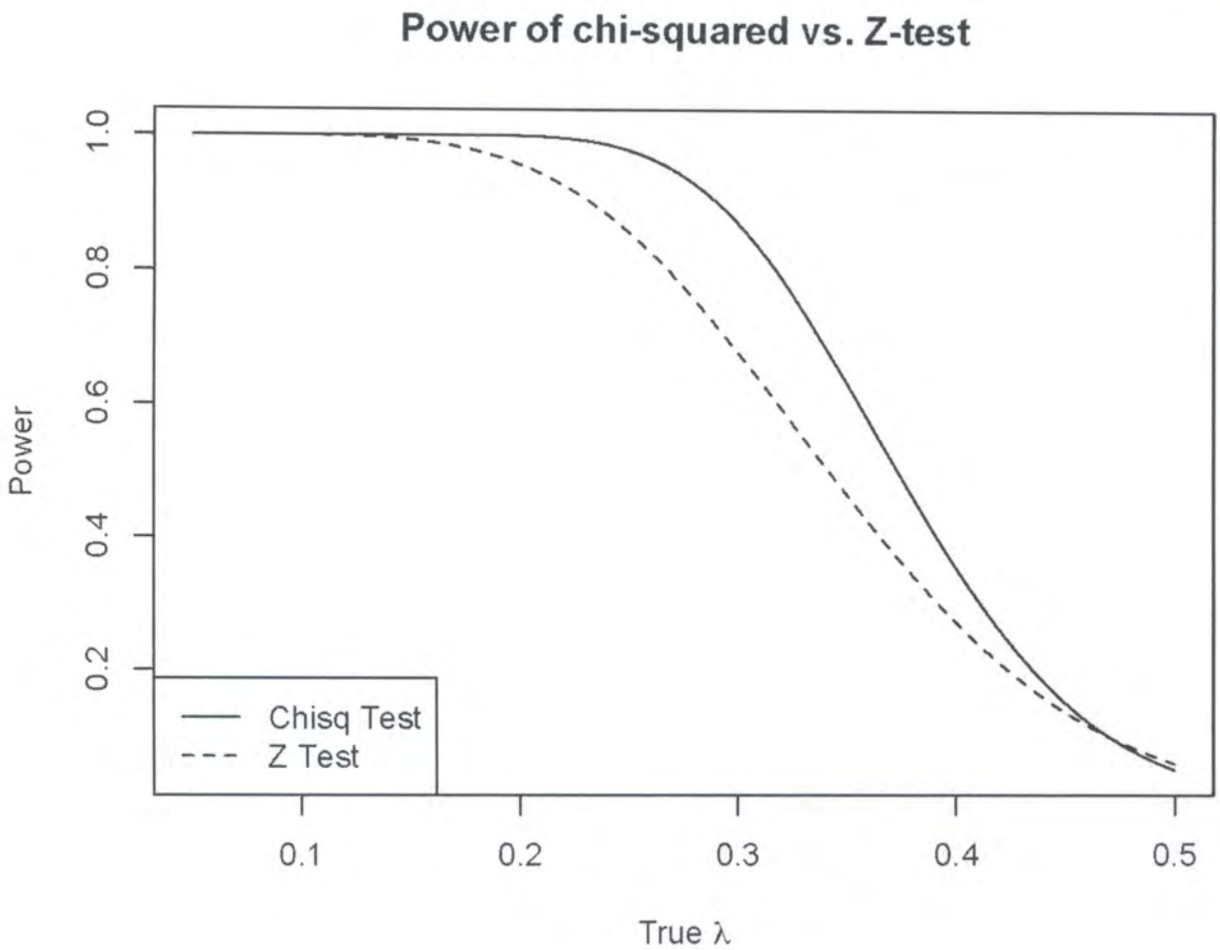


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# Compare those power functions
n = 29
lambda = seq(from=0.05,to=1/2,length=100)
Power = 1-pgamma(2.65,n,n*lambda)
powZ = 1-pgamma(2+1.645/(lambda*sqrt(n)),n,n*lambda)
xlabel = expression(paste("True ",lambda))
plot(lambda,Power,type='l',xlab=xlabel)
lines(lambda,powZ,lty=2)
title('Power of chi-squared vs. Z-test')
legend("bottomleft",lty=c(1,2),legend=c("Chisq Test","Z Test"))
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Incomplete example

$$Y_i = \beta x_i + E_i \qquad E_1 \dots E_n \overset{iid}{\sim} N(0, \sigma^2)$$

known

$$Y_i \sim N(\beta x_i, \sigma^2)$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}$$ Least squares & MLE

Does dosage level affect response?

$$H_0 : \beta = \beta_0 \quad H_1 : \beta \neq \beta_0$$

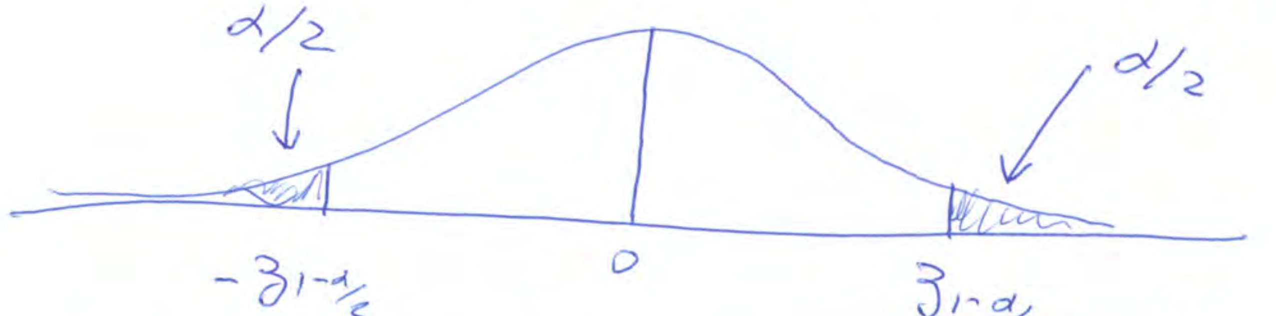
etc

test statistic $$Z = \frac{\sqrt{\sum_{i=1}^{n} x_i^2}\,(\hat{\beta} - \beta_0)}{\sigma}$$

$$\Omega_0 = \{\beta \in \mathbb{R} : \beta = \beta_0\}$$

$$\Omega_1 = \{\beta \in \mathbb{R} : \beta \neq \beta_0\}$$

Decision Rule: Reject $H_0$ if $|Z| \geq z_{1-\alpha/2}$




2-Sided test or 2-tailed test

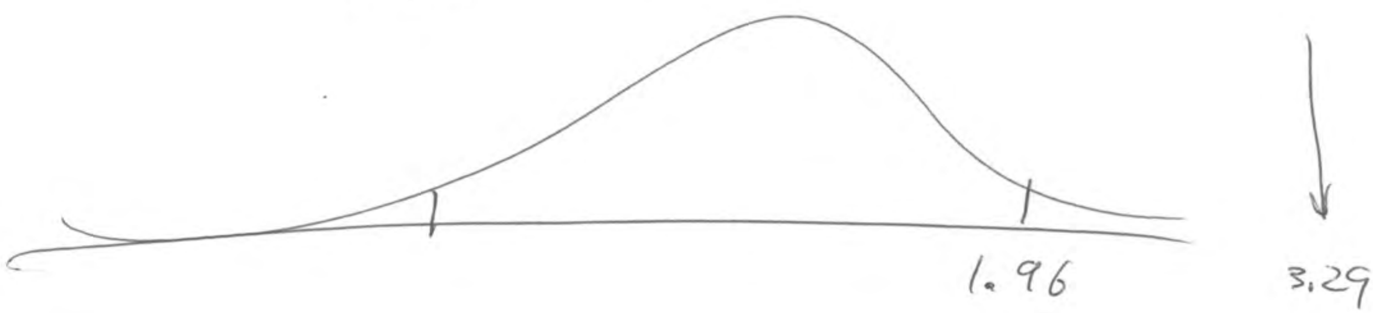
Observe data, get

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = 0.482$$

$$\sigma^2 = 0.6$$

$$Z = \frac{\sqrt{\Sigma x_i^2}\,(\hat{\beta} - 0)}{\sigma}$$

$$= \frac{\sqrt{28}\,(0.482 - 0)}{\sqrt{.6}} = 3.29$$




Conclusion Reject $H_0 : \beta = 0$

Do we conclude $\beta \neq 0$

or $\beta > 0$ ?