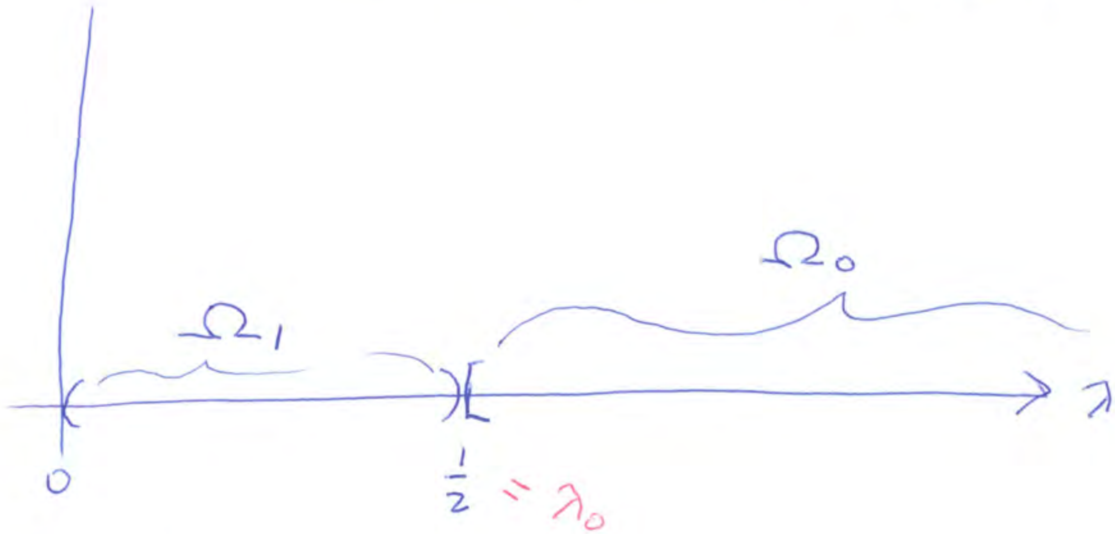
Tuesday Feb 25th

Last week was reading week

Model for surgery wait times:

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Exponential}(\lambda)$ $H_0: \lambda \geq \frac{1}{2}$ vs $H_1: \lambda < \frac{1}{2}$



Found $\bar{X}_n \sim \text{Gamma}(\alpha = n, \lambda' = n\lambda)$

Reject H_0 when $\bar{X}_n \geq k$

Want test with significance level α.

Find k so that $\max\limits_{\lambda \in \Omega_0} P_\lambda(\bar{X}_n \geq k) = \alpha$

Maximize $P_\lambda(\bar{X}_n \geq k)$ over $\Omega_0 = [\lambda_0, \infty)$

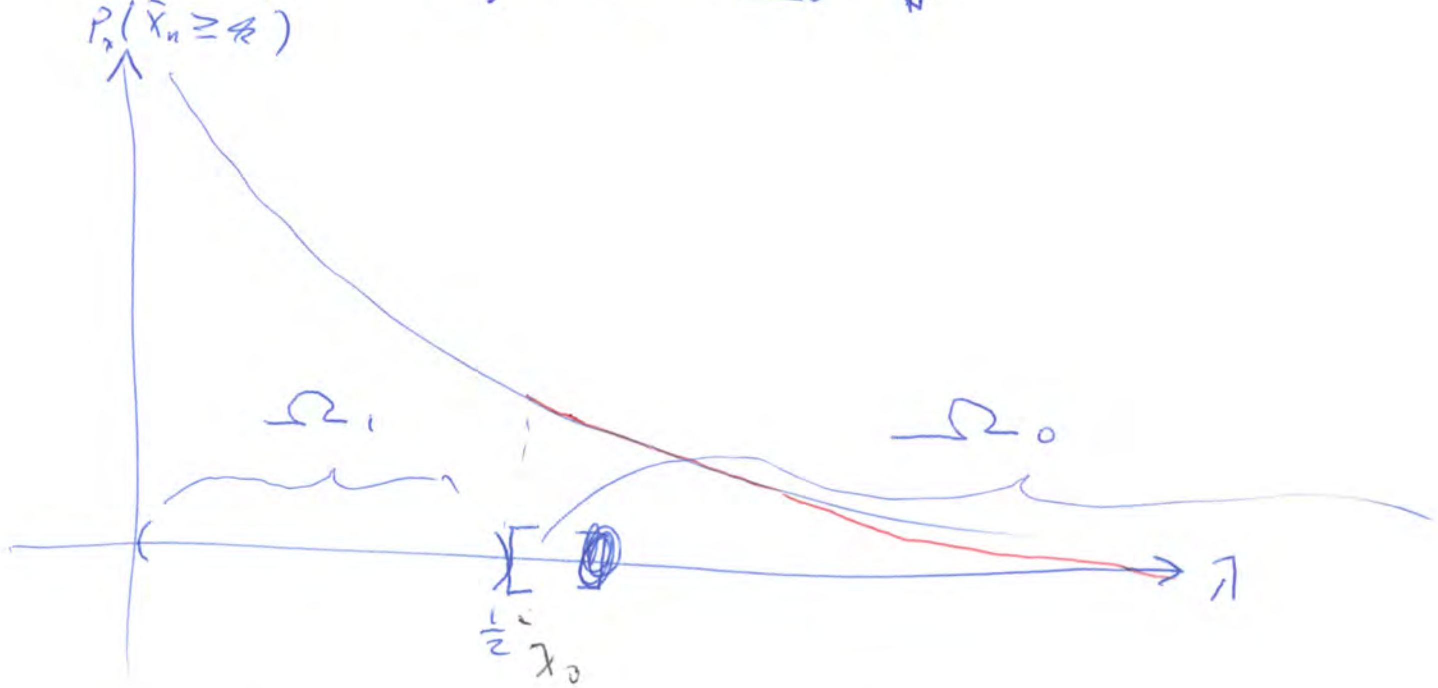
Since $\bar{X}_n \sim \text{Gamma}(\alpha = n, \lambda n)$

$$P_\lambda(\bar{X}_n \geq k) = \int_k^\infty \frac{(n\lambda)^n}{\Gamma(n)} e^{-n\lambda x} \bar{x}^{n-1}\, d\bar{x}$$

$$= \overset{\text{Last time}}{\cdots} = 1 - F_W(n\lambda k), \text{ where } W \sim G(n, 1)$$

$$\frac{d}{d\lambda} P_\lambda(\bar{X}_n \geq k) = \frac{d}{d\lambda}\left[1 - F_W(n\lambda k)\right]$$

$$= -f_W(n\lambda k) \cdot nk \quad \underline{\text{neg}} \downarrow$$



max over Ω_0 at $\lambda = \lambda_0 = \frac{1}{2}$

Comments

- This is why boundary point was in Ω_0
- With $H_0: \lambda = \lambda_0 = \frac{1}{2}$ vs $H_1: \lambda < \lambda_0$
- Nicer if $\Omega_0 \cup \Omega_1 = \Omega$

Def A hypothesis (null or alternative) (12)
that completely specifies the distribution of the sample data is called a SIMPLE H_0. Otherwise, COMPOSITE.

Anyway Need $P_{\lambda_0}(\bar{X}_n \geq k) = \alpha$

Consider

$$M_{\bar{X}_n}(t) = \left(1 - \frac{t}{\lambda_0 n}\right)^{-n} \quad \text{and}$$

$$M_Y(t) = (1-2t)^{-r/2} \quad \chi^2 \text{ MGF}$$

Let $Y = 2n\lambda_0 \bar{X}_n$ will be test statistic

$$M_Y(t) = M_{2n\lambda_0\bar{X}}(t) = M_{\bar{X}}(2n\lambda_0 t)$$

$$= \left(1 - \frac{2n\lambda_0 t}{n\lambda_0}\right)^{-\frac{2n}{2}}$$

$$= (1-2t)^{-\frac{2n}{2}} \quad \text{MGF of } \chi^2(2n)$$

$$P_{\lambda_0}(\bar{X}_n \geq k) = P_{\lambda_0}(2n\lambda_0 \bar{X}_n \geq 2n\lambda_0 k)$$

$$= P_{\lambda_0}(Y \geq \underbrace{2n\lambda_0 k}_{\text{set } = \chi^2_{1-\alpha}(2n)})$$

Decision Rule Reject H_0 if

$$Y = 2n\lambda_0 \bar{X}_n \geq \chi^2_{1-\alpha}(2n)$$

Packaged

Model $X_1 \ldots X_n \overset{iid}{\sim} Exp(\lambda)$

$H_0: \lambda \geq \lambda_0$ vs $H_1: \lambda < \lambda_0$

under the null hypothesis, when H_0 is true

Test statistic: $Y = 2n\lambda_0 \bar{X}_n \overset{H_0}{\sim} \chi^2(2n)$

Decision Rule Reject H_0 if $Y \geq \chi^2_{1-\alpha}(2n)$

Give the critical value(s).

Example $n=29,\ \bar{x}=2.5,\ s^2=3.6$

~~Give the~~ Calculate the test statistic; show a little work. The answer is a number.

$$y = 2n\lambda_0\bar{x} = 2*29*\tfrac{1}{2}*2.5 = 72.5$$

~~What~~ Give the critical value(s) at $\alpha=0.05$. Your answer is one number or two numbers.

$$df = 2*29 = 58$$

df	$\chi^2_{0.95}$
50	67.5
60	79.08

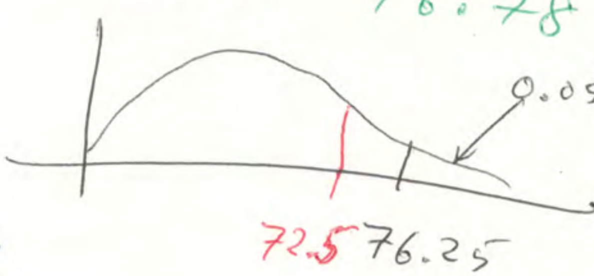
Interpolation

80%

$$67.5 + 0.8(79.08 - 67.5) = 76.25$$

Exact value is 76.78

$$Y = 72.5 < 76.25$$



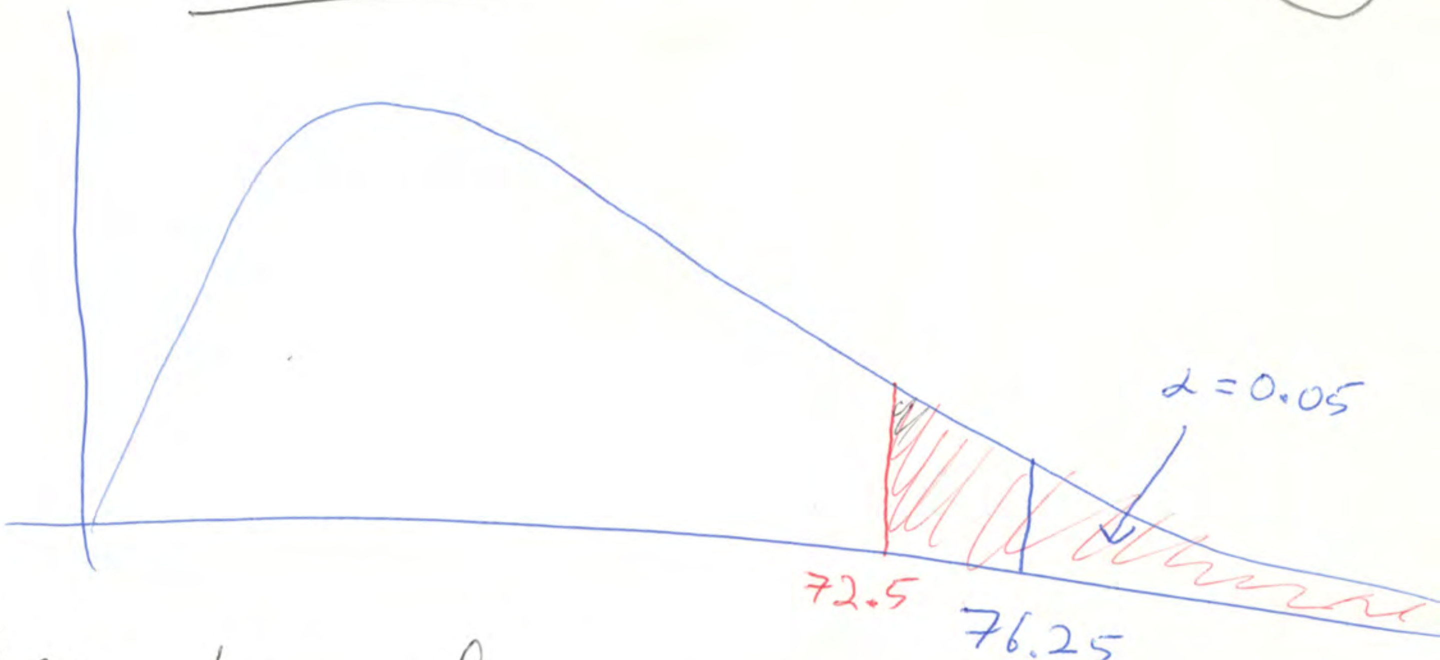
Decision: Don't Reject H_0

Conclusion; Pick one $\lambda \geq \frac{1}{2}$

$\lambda < \frac{1}{2}$

Is there evidence that the hospital is violating the regulation? No

P-values



Area beyond the observed test statistic, in direction of H_1

is The smallest sig level at which H_0 could be rejected.

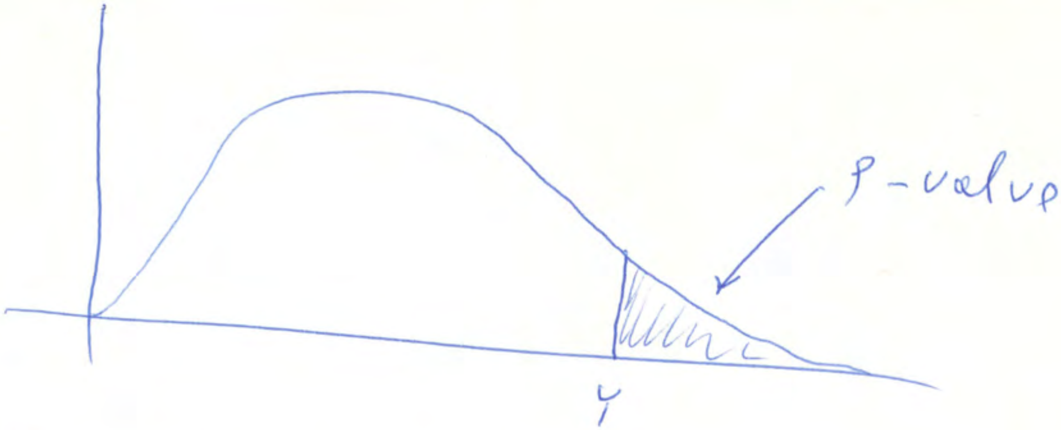
Def

P-value

Probability of getting results as surprising or more surprising than results we have observed, if H_0 is true

Reject H_0 iff $p < \alpha$.

Call the results "statistically significant"



$$p\text{-value} = 1 - F_Y(Y \mid \lambda = \lambda_0)$$

$$= 1 - F_Y(72.5 \mid \lambda = \tfrac{1}{2})$$

$$\overset{R}{=} \cancel{1 - \text{pgamma}(72.5,}$$

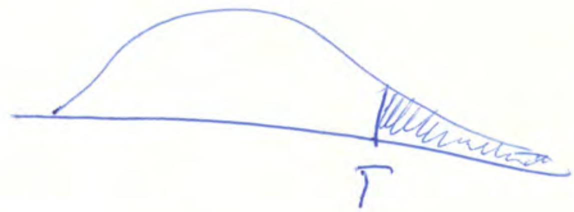
$$1 - \text{pchisq}(72.5, df = 58)$$

$$= 1 - 0.905 = 0.095 > 0.05$$

Side excursion

Test statistic T, Reject H_0 for large values of T $\quad T \overset{H_0}{\sim} F_0$

p-value $\quad P = 1 - F_0(T)$



Is a random variable.

Find distribution of p-value when H_0 is true.

Note $\quad 0 \le P \le 1$

For $0 < x < 1$ $\quad f_P(x) =$

$$\frac{d}{dx} P(P \le x) = \frac{d}{dx} P\left(1 - F_0(T) \le x\right)$$

$$= \frac{d}{dx} P\left(F_0(T) \ge 1 - x\right) = \frac{d}{dx} P\left(F_0^{-1}(F_0(T)) \ge F_0^{-1}(1-x)\right)$$

$$= \frac{d}{dx} P\left(T \ge F_0^{-1}(1-x)\right)$$

$$= \frac{d}{dx} \left(1 - P\left(T < F_0^{-1}(1-x)\right)\right)$$

$$= -\frac{d}{dx} F_0\left(F_0^{-1}(1-x)\right) = -\frac{d}{dx}(1-x)$$

$$\frac{dx}{dx} = 1 \quad \text{for } 0 < x < 1$$ UNIFORM

Back to expected wait time

Another possible test statistic

Wait times $X_1, \dots, X_n \overset{iid}{\sim} Exp(\lambda)$

Mandate is $E(X_i) = \mu = \frac{1}{\lambda} \le 2$

Still like to Reject H_0 if $\bar{X}_n \ge 2$

Test statistic

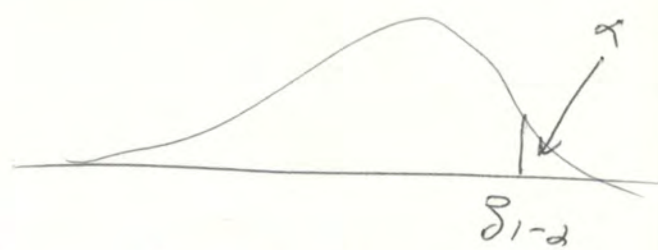
$$Z_n = \frac{\sqrt{n}\,(\bar{X}_n - \overset{2}{\mu_0})}{\hat{\sigma}_n} \overset{H_0}{\sim} N(0,1)$$

By CLT

$H_0: \mu \le \overset{2}{\mu_0}$ vs $H_1: \mu > 2$

Decision rule

Reject if $Z_n > z_{1-\alpha}$



Give the critical value for $\alpha = 0.05$ 1.96

$\alpha = 0.01$ 2.577

Which z-test

If you believe exponential model,

$E(X_i) = \frac{1}{\lambda}$ $\quad \hat{\lambda} = \frac{1}{\bar{x}}$

$Var(X_i) = \frac{1}{\lambda^2}$ $\quad \hat{\sigma}_n^2 = \bar{x}^2$

(19)

a) $$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\bar{X}_n} \xrightarrow{d} Z \sim N(0,1)$$

b) $$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{s_n} \xrightarrow{d} Z \sim N(0,1)$$

b is better if you're not sure of exponential model

If you are sure, probably $Y = 2n\lambda_0 \bar{X}_n$ is better.

Compute 3 test statistics on data
$n = 29$, $\bar{x} = 2.5$, $s^2 = 3.6$

1) $Y = 2n\lambda_0 \bar{x} = 72.25$, $p = 0.095$

a) $$Z_n = \frac{\sqrt{n}(\bar{X}_n - 2)}{\bar{X}_n} = 1.077$$

p-value = 

$\overset{\text{table}}{=} 0.1401$

b) $$Z_n = \frac{\sqrt{n}(\bar{x} - 2)}{s} = 1.42, \; p = 0.0778$$

If $H_0 : \mu = 2 \iff \lambda = \frac{1}{2}$ is true

what's $P(Z_n > 1.645)$ For test (b)

Suppose s^2 exactly equals σ^2 (impossible)

$\sigma^2 = \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{2}\right)^2} = 4$ $\quad \sigma = 2$

$P(Z \geq z_{1-\alpha}) = P\left(\frac{\sqrt{n}(\bar{X}-2)}{2} > 1.645\right)$

$= \cdots P(\bar{X} > 2.611)$

$= 1 - \text{pgamma}(2.611, \alpha = n, \lambda_0 * n)$

$= 1 - 0.941$

$= 0.059$