Thursday Feb. 13th

- Data $\underset{\sim}{x} \in S$ or $\underset{\sim}{X} \in S$, like $\underset{\sim}{X} = (X_1, \dots, X_n)$
- Model $\longrightarrow \{f_\theta : \theta \in \Omega\}$ or $\{P_\theta : \theta \in \Omega\}$
- Null hypothesis $H_0 : \theta \in \Omega_0$ vs $\underbrace{H_1 : \theta \in \Omega_1}_{\text{Alternative hypothesis}}$
- Critical Region $C \subseteq S$ Reject H_0 iff $x \in C$

Usually define C in terms of a test statistic $T(X)$; reject H_0 when T is greater than some critical value

- Significance Level $\alpha = \underset{\theta \in \Omega_0}{Max} P_\theta(X \in C)$

Usually $\alpha = 0.05$ or $\alpha = 0.01$

TRUTH

CONCLUSION		Null True $\theta \in \Omega_0$	Alternative True $\theta \in \Omega_1$
	$\theta \in \Omega_0$ conclude Null $x \notin C$	Correct	Type II Error
	$\theta \in \Omega_1$ conclude Alternative $X \in C$	TYPE I ERROR	Correct

6

Why would we want to reject H_0? If the data would be very surprising if H_0 were true.

Ex Cancer drug

Ex ~~Can~~ Ice cream

Development of a test

Example Government mandate: Average wait time for a type of surgery may not exceed 2 months.

Draw a random sample of patients, assess wait times. Are we in compliance.

Believe wait times are exponential (λ)

Idea of null Hypothesis: Everything is okay, no action required

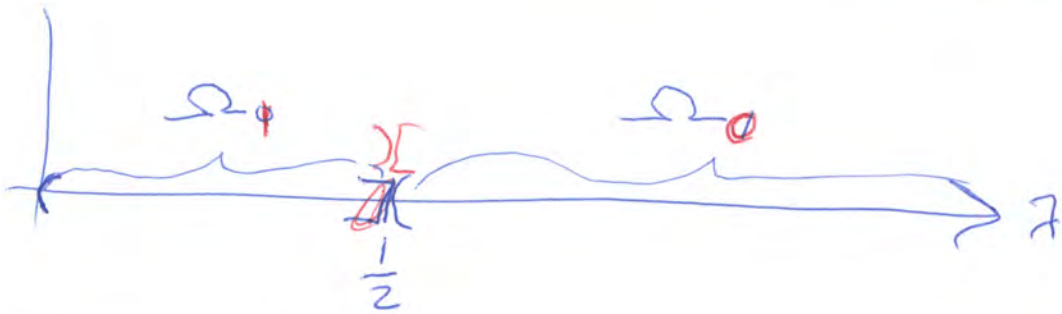
Alternative: Wait times too long

$$$

Model: $X_1 \dots X_n \overset{iid}{\sim} Exp(\lambda)$

H_0: $E(X_i) \leq 2 \Leftrightarrow \frac{1}{\lambda} \leq 2 \Leftrightarrow \lambda \geq 1/2$

H_1: $\lambda < 1/2$



Find a test by intuition; other ways later. If H_0 is true, a large $\bar{X}_n$ would be surprising.

Critical region: $C_k = \{ \underset{\sim}{x} \in \mathbb{R}^n : \bar{x}_n \geq k \}$ some k

Task: Choose k so $\max_{\theta \in \Omega_0} P_\theta(X \in C_k) = \alpha$

We want a "size α" test

(A good test statistic will emerge)

Distribution Theory (about $\bar{X}_n$)

$$M_{\bar{X}}(t) = M_{\frac{1}{n}\sum_{i=1}^{n} X_i}(t) = M_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right) \overset{\text{ind}}{=} \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right)$$

$$= \prod_{i=1}^{n}\left(1 - \frac{t}{\lambda n}\right)^{-1} = \left(1 - \frac{t}{\lambda n}\right)^{-n}$$

MGF of Gamma$(\alpha = n, \lambda = n\lambda)$

Want $\max_{\lambda \in \Omega_0} P_\lambda(\bar{X}_n \geq k) = \max_{\lambda : \lambda \geq \frac{1}{2}} P_\lambda(\bar{X}_n \geq k) = \alpha$

Need to maximize the probability over Ω_0.

Might feel stuck: $\frac{d}{d\lambda}\left(1 - F_{\bar{X}}(k \mid \lambda)\right)$

$$P_\lambda(\bar{X}_n \geq k) = \int_k^\infty \frac{(\lambda n)^n}{\Gamma(n)} e^{-n\lambda \bar{x}} \bar{x}^{n-1} \, d\bar{x}$$

Set $u = n\lambda\bar{x} \iff \bar{x} = \frac{u}{n\lambda}$ $\quad d\bar{x} = \frac{1}{n\lambda} du$

$\bar{x}$	u
∞	∞
k	$n\lambda k$

$$= \int_{n\lambda k}^{\infty} \frac{\cancel{(\lambda n)^{n}}}{\Gamma(n)} e^{-u} \left(\frac{u}{\cancel{n\lambda}}\right)^{n-1} \frac{1}{\cancel{n\lambda}}\, du$$

$$= \int_{n\lambda k}^{\infty} \underbrace{\frac{1}{\Gamma(n)} e^{-u} u^{n-1}\, du}_{GAMMA\ (\alpha = n,\ \lambda = 1)}$$

$$= 1 - F_W(n\lambda k) \quad , \text{ where } W \sim G(\alpha = n, \lambda = 1)$$

Decreasing in λ becaus

$$\frac{d}{d\lambda}\Big(1 - F_W(n\lambda k)\Big) = 0 - f_W(n\lambda k)\, nk$$

neg $\downarrow$