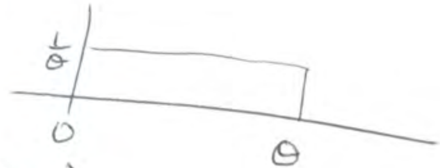
Tuesday Feb 11 (Part A)

Ex $X_1, \ldots, X_n \overset{iid}{\sim}$ Uniform $(0,\theta)$



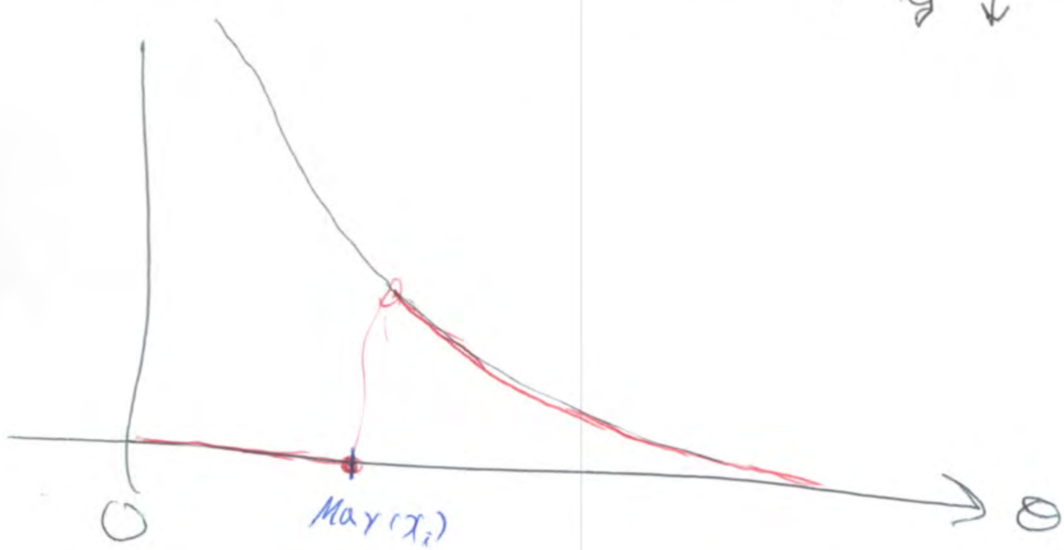
$$L(\theta) = L(\theta, \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 < x_i < \theta)$$

$$= \frac{1}{\theta^n} \prod_{i=1}^{n} I(0 < x_i < \theta)$$

Recall $I_{(A \cap B)}(x) = I_A(x) I_B(x)$

$$= \underbrace{\frac{1}{\theta^n}}_{\text{Decreasing because}} I(\mathrm{Max}(x_i) < \theta)$$

$$\frac{d}{d\theta} \theta^{-n} = (-n)\theta^{-n-1} \quad \text{neg} \downarrow$$



So MLE occurs at maximum data value: $\hat{\theta} = \mathrm{Max}(X_i)$

Ex $X_1 \dots X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \lambda)$

MOM: $E(X_i) = \mu = \frac{\alpha}{\lambda}$, $\text{var}(X_i) = \sigma^2 = \frac{\alpha}{\lambda^2}$

Set $\bar{X}_n = \frac{\hat{\alpha}}{\hat{\lambda}}$, $S^2 = \frac{\hat{\alpha}}{\hat{\lambda}^2}$

$$\frac{\bar{X}^2}{S^2} = \frac{\hat{\alpha}^2 / \cancel{\hat{\lambda}^2}}{\hat{\alpha} / \cancel{\hat{\lambda}^2}} = \hat{\alpha}$$

$$\frac{\bar{X}}{S^2} = \frac{\hat{\alpha}/\hat{\lambda}}{\hat{\alpha}/\hat{\lambda}^2} = \frac{\cancel{\hat{\alpha}}}{\hat{\lambda}} \cdot \frac{\hat{\lambda}^2}{\cancel{\hat{\alpha}}} = \hat{\lambda}$$

MLE $L(\alpha, \lambda) = \prod_{i=1}^{n} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x_i} x_i^{\alpha - 1}$

$$= \frac{\lambda^{n\alpha}}{\Gamma(\alpha)^n} e^{-\lambda \sum_{i=1}^n x_i} \left(\prod_{i=1}^n x_i\right)^{\alpha - 1}$$

$$\ell(\alpha, \lambda) = n\alpha \ln \lambda - n \ln \Gamma(\alpha) - \lambda \sum_{i=1}^n x_i + (\alpha - 1) \sum_{i=1}^n \ln x_i$$

$$\frac{\partial \ell}{\partial \alpha} = n \ln \lambda - n \frac{\frac{\partial}{\partial \alpha} \Gamma(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^n \ln x_i$$

$\ell(\alpha, \hat{\lambda}) = n\alpha \ln \lambda - n \ln \Gamma(\alpha) - \lambda \sum x_i + (\alpha - 1)\sum_{i=1}^{n} \ln x_i$ 15

Minimize Gamma Minus Log Likelihood

```
# Define the minus log likelihood function: Minimize this.
mll = function(theta,x)
     {
     alpha = theta[1]; lambda = theta[2]
     n = length(x)
     value = - n*alpha*log(lambda) + n*lgamma(alpha) +
               lambda*sum(x) - (alpha-1)*sum(log(x))
     return(value)
     } # End of function mll
```

Data are in GammaData

```
>
> # Calculate MOM estimates as starting values
> xbar = mean(GammaData); s2 = var(GammaData)
> alphaMOM = xbar^2/s2; lambdaMOM = xbar/s2
> startvals = c(alphaMOM,lambdaMOM)
> startvals # Take a look
[1] 5.232936 3.155775
>
> # The option method = 'L-BFGS-B' is necessary to put
> # bounds on the search.

> search1 = optim(par=startvals, fn = mll,
+            method = 'L-BFGS-B', lower = c(0,0), x = GammaData)
> search1
$par
[1] 5.211516 3.142857

$value
[1] 1032.146

$counts
function gradient
       7        7

$convergence
[1] 0

$message
[1] "CONVERGENCE: REL_REDUCTION_OF_F <= FACTR*EPSMCH"

> cat("MLE is alphahat, lambdahat = ", search1$par, "\n")
> cat("Truth is alpha=5, lambda=3 \n")

MLE is alphahat, lambdahat =  5.211516 3.142857
Truth is alpha=5, lambda=3
```

	LECTURE	BOOK
RANDOM VARIABLES	X	Could be x
DENSITIES PMFs	$f(x\mid\theta)$ or $p(x\mid\theta)$ with indicators for support	$f_\theta(x)$ sometimes for discrete distributions no
PARAMETER SPACE	$\theta \in \Omega$	$\theta \in \Omega$ ✓
MODEL	Set of assertions that directly or indirectly imply a probability distribution for the sample data. For Ex • $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ • $Y_i = \beta_0 + \beta_1 x_i + E_i$, where x_i are constants $E_1 \dots E_n \overset{iid}{\sim} N(0, \sigma^2)$	$\{P_\theta : \theta \in \Omega\}$ or $f_X \in \{f_\theta, \theta \in \Omega\}$
SAMPLE SPACE		

Lecture	Text
SAMPLE SPACE From STA 256 $S \xrightarrow{\underset{\sim}{X}} \mathbb{R}^n$ underlying sample space — sample space $\underset{\sim}{X}(s) = (X_1(s), X_2(s), \dots, X_n(s))$ Prob measure on subsets of S induced a prob measure on subsets of $\mathbb{R}^n$, like $P(X \le x) = P(\{s \in S : X(s) \le x\})$	$s \in S$ data — sample space $s = (x_1, \dots x_n)$ or $(X_1, \dots X_n)$ Joint distribution of $X_1, \dots X_n$ gives prob on subsets of $\mathbb{R}^n = S$.
Likelihood Function $L(\theta)$ or $L(\theta, x)$ $\ell(\theta)$	$L(\theta \mid s)$ <u>good</u> $\ell(\theta \mid s)$
Score function	$S(\theta \mid s) = \dfrac{\partial \ell(\theta \mid s)}{\partial \theta}$
α	$\gamma = 1 - \alpha$