Thurs. Feb. 6th

(Tuesday was Test one)

$$L(\theta) = L(\theta, x) = \prod_{i=1}^{n} f(x_i|\theta) \text{ or } \prod_{i=1}^{n} p(x_i|\theta)$$

$$\ell(\theta) = \ln L(\theta)$$

<u>Normal Example</u> $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\ell(\mu, \sigma^2) = -\frac{n}{2}\ln(\sigma^2) - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \ell}{\partial \mu} \overset{set}{=} 0 \implies \sum_{i=1}^{n} x_i = n\mu \qquad (1)$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 \frac{\partial}{\partial \sigma}(\sigma^2)^{-1}$$

$$= \frac{-n}{2\sigma^2} - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2(-1)(\sigma^2)^{-2}$$

$$= \frac{-n}{2\sigma^2} + \frac{1}{2}\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\sigma^4} \overset{set}{=} 0$$

$$\implies \frac{n}{\cancel{2}\cancel{\sigma^2}} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{\cancel{2}\cancel{\sigma^2}\sigma^2}$$

$$\implies \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \qquad (2)$$

From (1), $\mu = \bar{x}$, sub into (2)

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{n-1}{n} S^2$$

Can get away without the multi-variable 2nd derivative test

Note the solution to (1): $\sum x_i = n\mu$ did not depend on σ^2

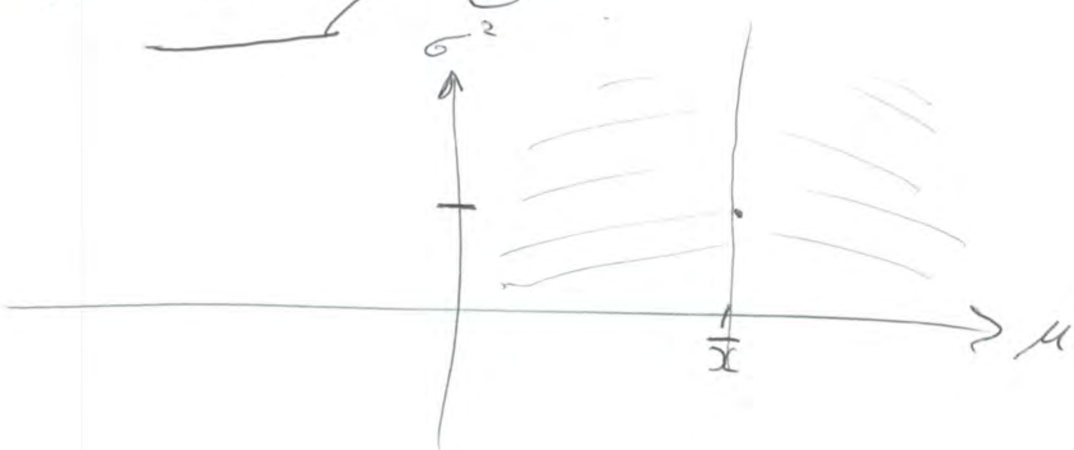
$$\cancel{\frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{\partial}{\partial \sigma^2}}$$

$$\frac{\partial^2 \ell}{\partial \mu^2} = \frac{\partial}{\partial \mu}\left(\frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i - n\mu\right)\right)$$

$$= \frac{1}{\sigma^2}(0 - n) = \frac{-n}{\sigma^2} \text{ neg}$$

CCD MAX

For every σ^2



$$\frac{\partial^2 \ell}{\partial(\sigma^2)^2} = \frac{\partial}{\partial \sigma^2}\left[-\frac{n}{2}[\sigma^2]^{-1} + \frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})^2(\sigma^2)^{-2}\right]$$

$$= + \frac{n}{2}(+1)[\sigma^2]^{-2} + \frac{1}{\cancel{2}}\sum_{i=1}^{n}(x_i-\bar{x})^2(-\cancel{2})(\sigma^2)^{-3}$$

$$= \frac{n}{2\sigma^4} - \frac{\sum(x_i-\bar{x})^2}{\sigma^6}$$

at $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \hat{\sigma}^2$,

$$\frac{n}{2\hat{\sigma}^4} - \frac{n\hat{\sigma}^2}{\hat{\sigma}^6} = \frac{n}{2\hat{\sigma}^4} - \frac{n}{\hat{\sigma}^4}$$

$$= \frac{n}{\hat{\sigma}^4}\left(\frac{1}{2}-1\right) = \frac{-n}{2\hat{\sigma}^4} < 0 \quad \text{CCD}$$

$\cap$ MAX

So MLE is

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Comments

1. $\hat{\sigma}^2$ is BIASED. Know S^2 is unbiased

$\hat{\sigma}^2 = \frac{n-1}{n} S^2$ ASYMPTOTICALLY UNBIASED

2. Could have differentiated w/resp (with respect to) to σ instead of σ^2, SAME RESULT

This is an example of INVARIANCE PRINCIPLE of maximum likelihood estimation, which says (ROUGHLY) the MLE of a function is that function of the MLE.

More precisely

Theorem If the likelihood function $L(\theta)$ has a unique maximum at $\theta = \hat{\theta}$ and $\alpha = g(\theta)$ is a one-to-one reparameterization, then

$$\hat{\alpha} = g(\hat{\theta}).$$

EX Normal: $g(\mu, \sigma^2) = (\mu, \sigma)$

EX Exponential $\theta = \frac{1}{\lambda}$, $E(X) = \theta$

Ex from text! Watch out

$X_1 \dots X_n \overset{ind}{\sim} N(\mu_i, 1) \qquad \underset{\sim}{\theta} = (\mu_1, \mu_2, \dots \mu_n)$

Want to estimate $\alpha = g(\underset{\sim}{\theta}) = \sum_{i=1}^{n} \mu_i^2$

How about $\hat{\alpha} = g(\vec{\theta})$

$$l(\theta) = \ln \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i - \mu_i)^2}$$

$$\ln (2\pi)^{-\frac{n}{2}} + \ln e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_i)^2}$$

$$-\frac{n}{2} \ln (2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_i)^2 \qquad NEG$$

MAKE 2nd term zero

$$\hat{\mu}_1 = x_1, \ \hat{\mu}_2 = x_2 \ \dots, \hat{\mu}_n = x_n$$

How about $\hat{\alpha} = \sum_{i=1}^{n} \hat{\mu}_i^2$ "Plug-in" estimator

$$= \sum_{i=1}^{n} X_i^2$$

Recall $\sigma^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$

$$\Rightarrow E(X^2) = \sigma^2 + \mu^2$$

$$E(\hat{\alpha}_n) = \sum_{i=1}^{n} E(X_i^2) = \sum_{i=1}^{n}(1 + \mu_i^2) = n + \sum_{i=1}^{n} \mu_i^2$$

$$= n + \alpha$$