Thurs. Jan. 23

Methods for Finding Estimators

- Method of Moments (MOM)
- Least squares
- Maximum Likelihood

Method of Moments

Population Moments	Sample Moments
$E(X)$	$\bar{X}_n$
$E(X^2)$	$\frac{1}{n}\sum_{i=1}^{n} X_i^2$
$Var(X) = E\{(X-\mu)^2\}$	$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2$ or S^2
$E(X^k)$	$\frac{1}{n}\sum_{i=1}^{n} X_i^k$
$E\{(X-\mu)^k\}$	$\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^k$

Sample moments $\xrightarrow{P}$ Pop. Moments

Recipe for MoM

(1) Write population moment(s) in terms of parameter(s).

(2) Substitute sample moments for pop. moments, put a hat on θ.

(3) Solve for $\hat{\theta}$

Ex $X_1, \dots X_n \overset{iid}{\sim} U(0, \theta)$ (1) $E_\theta(X) = \frac{\theta}{2}$

(2) $\bar{X}_n = \frac{\hat{\theta}}{2}$ (3) $\hat{\theta} = 2\bar{X}_n$

If $E_\theta(X) = \theta$, take $\hat{\theta} = \bar{X}_n$

- Bernoulli
- Normal
- Poisson
- χ^2

Ex Let $X_1, \dots X_n \overset{iid}{\sim}$ Geometric(θ)

(1) $E_\theta(X) = \frac{1-\theta}{\theta}$ (2) $\bar{X}_n = \frac{1-\hat{\theta}}{\hat{\theta}} = \frac{1}{\hat{\theta}} - 1$

$\iff \frac{1}{\hat{\theta}} = 1 + \bar{X}_n \iff \hat{\theta} = \frac{1}{1+\bar{X}_n}$

Ex $X_1 \ldots X_n \overset{iid}{\sim} \text{Gamma}(\alpha, \lambda)$

1. $E(X) = \frac{\alpha}{\lambda}$ $\quad Var(X) = \frac{\alpha}{\lambda^2}$

2. $\bar{X}_n = \frac{\hat{\alpha}}{\hat{\lambda}}$ $\quad S^2 = \frac{\hat{\alpha}}{\hat{\lambda}^2}$ Solve

$$\frac{\bar{X}_n^2}{S^2} = \frac{\hat{\alpha}^2 / \cancel{\hat{\lambda}^2}}{\hat{\alpha} / \cancel{\hat{\lambda}^2}} = \hat{\alpha}$$

$$\bar{X}_n = \frac{\hat{\alpha}}{\hat{\lambda}} \iff \hat{\lambda} = \frac{\hat{\alpha}}{\bar{X}_n} = \frac{\bar{X}_n^2}{S^2 \bar{X}_n} = \frac{\bar{X}_n}{S^2}$$

Have $\hat{\alpha} = \frac{\bar{X}_n^2}{S^2}$, $\hat{\lambda} = \frac{\bar{X}_n}{S^2}$

Ex $X_1 \ldots X_n \overset{iid}{\sim} U(-\theta, \theta)$

1. $E_\theta(X) = 0$ $\quad$ oops $\quad E(X^2) = \int_{-\theta}^{\theta} x^2 \frac{1}{2\theta}\, dx$

$$= \frac{1}{2\theta} \int_{-\theta}^{\theta} x^2\, dx = \frac{1}{2\theta} \left. \frac{x^3}{3} \right|_{-\theta}^{\theta}$$

$$= \frac{1}{6\theta}\left(\theta^3 - (-\theta)^3\right) = \frac{2\theta^3}{6\theta} = \frac{\theta^2}{3}$$

2. $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\hat{\theta}^2}{3} \iff \hat{\theta}^2 = \frac{3}{n}\sum_{i=1}^{n} X_i^2$

$$\iff \hat{\theta} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} X_i^2}$$

Note: There can be can be more than one "right" answer

Ex $X_1 \ldots X_n \overset{iid}{\sim}$ Poisson(λ) $E(X) = Var(X) = \lambda$

So $\bar{X}_n$ & S^2 both MOM

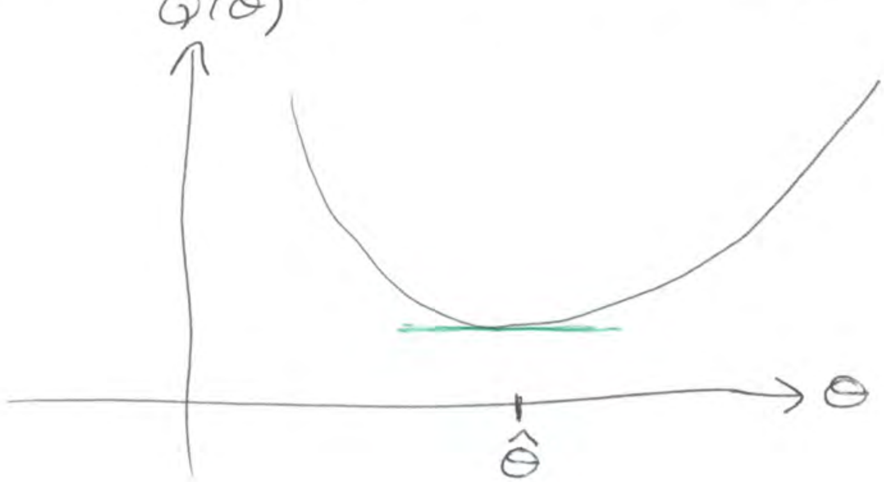
Prefer $\bar{X}_n$, in general Lower "order" moments

Method of Least Squares

Idea: Guess a value of θ that makes the observed data as close as possible to what you'd expect.

Minimize the sum of squares

$$Q(\theta) = \sum_{i=1}^{n} \left(X_i - E_\theta(X_i) \right)^2$$



We are "Learning" about θ by minimizing a "Loss function."

Ex Spse $\theta = E_\theta(X) = \mu$

$$Q(\mu) = \sum_{i=1}^{n} (X_i - \mu)^2$$

$$\frac{dQ}{d\mu} = \frac{d}{d\mu} \sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} \frac{d}{d\mu} (X_i - \mu)^2$$

$$= \cancel{\sum_{i=1}^{n} 2 (X_i - \mu) d = 2 \sum_{i=1}}$$

$$= \sum_{i=1}^{n} 2 (X_i - \mu) \frac{d}{d\mu} (X_i - \mu) = 2 \sum_{i=1}^{n} (X_i - \mu)(-1)$$

$$= -2 \sum_{i=1}^{n} (X_i - \mu) = -2 \left[\sum_{i=1}^{n} X_i - n\mu \right] \stackrel{\text{set}}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} X_i = n\mu \Rightarrow \mu = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}_n$$

$$\hat{\mu} = \bar{X}_n$$

(1) $\bar{X}_n$ minimizes sum of squares for <u>any</u> set of numbers

(2) LS = MOM