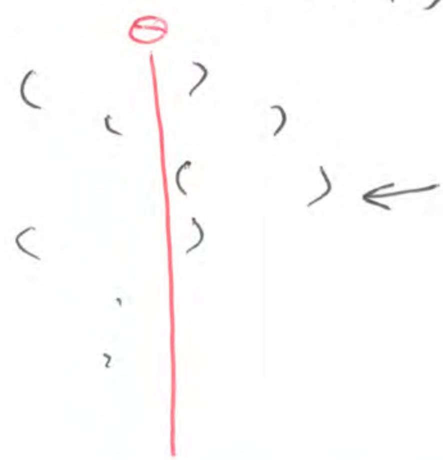
Thurs. Jan 16

Clean up from last time

Had 95% CI $\bar{X}_n \pm 1.96 \frac{\sqrt{\bar{X}_n(1-\bar{X})}}{\sqrt{n}}$

$\rightarrow (0.636, 0.764)$ Does this mean

$P(0.636 < \theta < 0.764)$? NO



Forget the Law of Total Prob stuff in B-Ball, &

Let $X_1 \cdots X_n \overset{iid}{\sim}$ Geometric(θ)

Spse $n = 50$, $\bar{x}_n = 5.68$

1) Give a point estimate of θ

(Method of moments example)

$$E(X_i) = \mu = \frac{1-\theta}{\theta} \Leftrightarrow \mu\theta = 1-\theta \Leftrightarrow 1 = \mu\theta + \theta$$

$$\Leftrightarrow 1 = \theta(1+\mu) \Leftrightarrow \theta = \frac{1}{1+\mu}$$

Put hat on θ, sub $\bar{X}_n$ for μ, estimate

$$\hat{\theta}_n = \frac{1}{1+\bar{X}_n}, \quad \hat{\theta} = \frac{1}{1+5.68} = 0.1497$$

2) Is estimator consistent Yes

$\mu = \frac{1-\theta}{\theta}$ By LLN, $\bar{X}_n \xrightarrow{P} \mu$. Since the function $g(x) = \frac{1}{1+x}$ is continuous for $x > 0$

So by continuous mapping,

$$g(\bar{X}_n) \xrightarrow{P} g(\mu) = \left(\frac{1}{1+\mu}\right) = \frac{1}{1+\frac{1-\theta}{\theta}}$$

$$= \frac{1}{\frac{\cancel{\theta}+1-\cancel{\theta}}{\theta}} = \theta \quad \text{consistent}$$

9

3) Is $\hat{\Theta}_n = \dfrac{1}{1+\bar{X}_n}$ unbiased?

$$E\left(\frac{1}{1+\bar{X}_n}\right) = ?$$

4) Give a 95% CI for Θ.

General method: Derive a CI for μ
convert it to CI for Θ

Start with CI for μ based on CLT

$$1-\alpha \approx P\left(\bar{X}_n - \frac{\hat{\sigma}_n}{\sqrt{n}} z_{1-\alpha/2} < \mu < \bar{X}_n + z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}\right)$$

Need consistent estimator of $\sigma^2 = \dfrac{1-\Theta}{\Theta^2}$

$$\hat{\sigma}_n^2 = \frac{1-\hat{\Theta}_n}{\hat{\Theta}_n^2} = \frac{1-\frac{1}{1+\bar{X}_n}}{\left(\frac{1}{1+\bar{X}_n}\right)^2} = \frac{\frac{1+\bar{X}_n-1}{1+\bar{X}_n}}{\frac{1}{(1+\bar{X}_n)^2}}$$

$$= \frac{\bar{X}_n}{1+\bar{X}_n}\cdot(1+\bar{X}_n)^2$$

$$= \bar{X}_n(1+\bar{X}_n)$$

$$1-\alpha \approx P\left(\bar{X}_n - z_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1+\bar{X}_n)}{n}} < \mu < \bar{X}_n + z_{1-\alpha/2}\sqrt{\frac{\bar{X}_n(1+\bar{X}_n)}{n}}\right)$$

$$= P(L < \mu < U) = P\left(L < \frac{1-\theta}{\theta} < U\right)$$

$\left(\begin{array}{l}\text{Want to isolate } \theta\text{. Previously showed} \\ g(\mu) = \dfrac{1}{1+\mu} = \theta \quad \text{Note } g(\mu) \downarrow\end{array}\right)$

$$= P(g(L) > g(\mu) > g(U))$$

$$= P(g(U) < \overset{\shortparallel}{\theta} < g(L))$$

For observed $\bar{x} = 5.68$ $\quad \ell = 5.68 - 1.96\sqrt{\frac{5.68(6.68)}{50}}$

$\ell = 5.68 - 1.71 = 3.97$

$u = 5.68 + 1.71 = 7.39$

CI for θ is $\left(\dfrac{1}{1+7.39}, \dfrac{1}{1+3.97}\right)$

$= (0.12, 0.20)$ ✓

Thurs Jan 16

Confidence intervals Part 2

$X_1, \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Estimate μ with $\bar{X}_n$, want CI

Have $\bar{X}_n \pm z_{1-\alpha/2} \frac{S}{\sqrt{n}}$ OKAY for large samples

Based on

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{S} \xrightarrow{d} Z \sim N(0,1)$$