Tues Jan 14

Continuous mapping If $g: \mathbb{R}^k \to \mathbb{R}^p$ is continuous at $\underset{\sim}{c}$

$$\underset{\sim}{X}_n \xrightarrow{P} \underset{\sim}{c} \quad \text{then } g(\underset{\sim}{X}_n) \xrightarrow{P} g(\underset{\sim}{c})$$

Law of Large numbers

Let $\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_n$ be independent random vectors with expected value $\underset{\sim}{\mu}$.

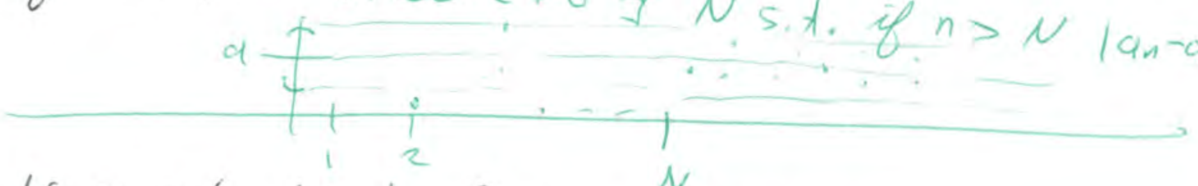
$$\overline{X}_n \xrightarrow{P} \underset{\sim}{\mu}$$

Degenerate RV $\quad P(X_n = a_n) = 1$

Theorem If the constants $a_1, a_2, \dots$ satisfy
$\lim_{n\to\infty} a_n = a$, then $a_n \xrightarrow{P} a$ As degenerate RVs.

Proof Seek to show For all $\varepsilon > 0$ $\lim_{n\to\infty} P\{|a_n - a| < \varepsilon\} = 1$

By def of limit For all $\varepsilon > 0$ $\exists$ N s.t. if $n > N$ $|a_n - a| < \varepsilon$



There is N such that for all $n > N$

$$|a_n - a| < \varepsilon \Rightarrow P(|a_n - a| < \varepsilon) = 1 \text{ And}$$

$$\lim P(|a_n - a| < \varepsilon) = \lim_{n\to\infty} 1 = 1 \qquad \blacksquare$$

(22)

$$S_n^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n-1} = \underbrace{\left(\frac{n}{n-1}\right)}_{a_n} \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \left(\frac{n}{n-1}\right) \cdot \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)$$

$$= \left(\frac{n}{n-1}\right) \cdot \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - 2\bar{X}_n \sum_{i=1}^{n} X_i + \sum_{i=1}^{n} \bar{X}_n^2\right)$$

$$= \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \frac{n\bar{X}_n^2}{\cancel{n}}\right)$$

$$= \left(\frac{n}{n-1}\right)\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2\right)$$

Since $\lim_{n\to\infty} \frac{n}{n-1} = 1$, $\frac{n}{n-1} \xrightarrow{P} 1$

By LLN $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X_i^2) = \sigma^2 + \mu^2$

$Var(X) = E(X^2) - (E(X))^2$

$\sigma^2 = E(X^2) - \mu^2$

$\Rightarrow E(X^2) = \sigma^2 + \mu^2$

$\bar{X}_n \xrightarrow{P} \mu$ by LLN & so by continuous mapping

$$\bar{X}_n^2 \xrightarrow{P} \mu^2$$

Finally, the function $g(x, y, z) = x(y - z)$ is continuous, so

$$S_n^2 = g\left(\frac{n}{n-1}, \frac{1}{n}\sum_{i=1}^{n} X_i^2, \bar{X}_n^2\right) \to g(1, \sigma^2 + \mu^2, \mu^2)$$

$$= 1 \cdot (\sigma^2 + \mu^2 - \mu^2) = \sigma^2$$ <u>consistent</u>