STA 260 Formulas and Tables

$$\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a} \qquad \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \qquad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt \qquad \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

$$I_A(x) \stackrel{def}{=} \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases} \qquad I(x \in A) \stackrel{def}{=} \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases} \qquad g(x)I(x \in A) = \begin{cases} g(x) & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

$$F_X(x) \stackrel{def}{=} P(X \le x) \qquad F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt \qquad F_X(x) = \sum_t I(t \le x)\, p_X(t)$$

$$p_X(x) = \sum_y p_{X,Y}(x,y) \qquad p_{Y|X}(y|x) \stackrel{def}{=} \frac{p_{X,Y}(x,y)}{p_X(x)} \qquad p_X(x) = \sum_y p_{X|Y}(x|y)\, p_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad f_{Y|X}(y|x) \stackrel{def}{=} \frac{f_{X,Y}(x,y)}{f_X(x)} \qquad f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy$$

$$E(X) \stackrel{def}{=} \sum_x x\, p_X(x) \qquad E(g(X)) = \sum_x g(x)\, p_X(x) \qquad E(g(X,Y)) = \sum_x \sum_y g(x,y)\, p_{X,Y}(x,y)$$

$$E(X) \stackrel{def}{=} \int_{-\infty}^{\infty} x\, f_X(x)\, dx \qquad E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \qquad E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$Var(X) \stackrel{def}{=} E\left((X-\mu)^2\right) \qquad Var(X) = E(X^2) - [E(X)]^2 \qquad Var(aX) = a^2 Var(X)$$

$$Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$$

$$Cov(X,Y) \stackrel{def}{=} E[(X-\mu_X)(Y-\mu_Y)] \qquad Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$Cov(aX, bY) = ab\, Cov(X,Y) \qquad Cov\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = \sum_{i=1}^n \sum_{j=1}^m Cov(X_i, Y_j)$$

$$Var\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i^2 Var(X_i) + \sum\sum_{i \ne j} a_i b_j Cov(X_i, X_j)$$

If $X_1, \ldots, X_n$ are independent, $Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i)$

$$M_X(t) \stackrel{def}{=} E(e^{Xt}) \qquad M_X^{(k)}(0) = E(X^k)$$

$M_{aX}(t) = M_X(at)$ $\qquad$ $M_{\sum X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$ if the X_i are independent.

Convergence in probability:

$T_n \xrightarrow{p} c$ means for all $\epsilon > 0$, $\lim_{n\to\infty} P\{|T_n - c| \geq \epsilon\} = 0 \Leftrightarrow \lim_{n\to\infty} P\{|T_n - c| < \epsilon\} = 1$

Variance rule: If $\lim_{n\to\infty} E(T_n) = c$ and $\lim_{n\to\infty} Var(T_n) = 0$, then $T_n \xrightarrow{p} c$.

Law of Large Numbers: $\overline{X}_n \xrightarrow{p} \mu = E(X_i)$.

Continuous mapping: If $T_n \xrightarrow{p} c$ and $g(x)$ is continuous at $x = c$, then $g(T_n) \xrightarrow{p} g(c)$

Convergence in Distribution:

$X_n \xrightarrow{d} X$ means $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at every point where $F_X(x)$ is continuous.

$\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ $\qquad$ $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$

Central Limit Theorem: If $X_1, \ldots, X_n \stackrel{ind}{\sim} ?(\mu, \sigma^2)$, then $Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim$ Normal (0,1).

If $X_1, \ldots, X_n \stackrel{ind}{\sim} ?(\mu, \sigma^2)$ and $\widehat{\sigma}_n^2 \xrightarrow{p} \sigma^2$, then $\frac{\sqrt{n}(\overline{X}_n - \mu)}{\widehat{\sigma}_n} \xrightarrow{d} Z \sim$ Normal (0,1).

$\overline{X}_n \pm z_{1-\alpha/2} \frac{\widehat{\sigma}_n}{\sqrt{n}}$ $\qquad$ $T = \frac{Z}{\sqrt{Y/\nu}} \sim t(\nu)$ $\qquad$ $\overline{X} \pm t_{1-\alpha/2} \frac{S}{\sqrt{n}}$

Distribution	$f_X(x\|\theta)$ or $p_X(x\|\theta)$	$M_X(t)$	$E(X)$	$Var(X)$
Bernoulli	$\theta^x(1-\theta)^{1-x} I(x = 0, 1)$	$\theta e^t + 1 - \theta$	θ	$\theta(1-\theta)$
Binomial	$\binom{m}{x}\theta^x(1-\theta)^{m-x} I(x = 0, \ldots, m)$	$(\theta e^t + 1 - \theta)^m$	$m\theta$	$m\theta(1-\theta)$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!} I(x = 0, 1, \ldots)$	$e^{\lambda(e^t - 1)}$	λ	λ
Geometric	$(1-\theta)^x\theta\, I(x = 0, 1, \ldots)$	$\theta\left(1 - (1-\theta)e^t\right)^{-1}$	$\frac{1-\theta}{\theta}$	$\frac{1-\theta}{\theta^2}$
Exponential	$\lambda e^{-\lambda x}\, I(x > 0)$	$(1 - \frac{t}{\lambda})^{-1}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma	$\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}\, I(x > 0)$	$(1 - \frac{t}{\lambda})^{-\alpha}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$
Chi-squared (χ^2)	$\frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-x/2} x^{\nu/2-1} I(x > 0)$	$(1 - 2t)^{-\nu/2}$	ν	2ν
Normal	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{\mu t + \frac{1}{2}\sigma^2 t^2}$	μ	σ^2
Uniform	$\frac{1}{R-L} I(L < x < R)$	$\frac{e^{Rt} - e^{Lt}}{t(R-L)}$	$\frac{L+R}{2}$	$\frac{(R-L)^2}{12}$
Beta	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\, I(0 < x < 1)$	Useless	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

If $X \sim$ Exponential (λ), $F_X(x) = 1 - e^{-\lambda x} I(x > 0)$. $\qquad$ If $X \sim$ N(μ, σ^2), $\frac{X-\mu}{\sigma} \sim$ N(0,1)

A *random sample* is a collection of independent and identically distributed random variables $X_1, \ldots, X_n$. Write $X_1, \ldots, X_n \stackrel{i.i.d}{\sim} P_\theta$, $\theta \in \Omega$. θ is the *parameter*, and Ω is the *parameter space*.

An estimator $T_n = T_n(X_1, \ldots, X_n)$ is said to be *unbiased* for θ if $E(T_n) = \theta$ for all $\theta \in \Omega$.

An estimator $T_n = T_n(X_1, \ldots, X_n)$ is said to be *consistent* for θ if $T_n \xrightarrow{p} \theta$ for all $\theta \in \Omega$.

Method of Moments: Substitute sample moments for population moments and put hats on.

Least squares: Minimize $Q(\theta) = \sum_{i=1}^n \left(Y_i - E_\theta(Y_i)\right)^2$

If $X_1, \ldots, X_n$ is a random sample from an Exponential(λ) distribution,

- $\overline{X}_n \sim \text{Gamma}(n, n\lambda)$.
- $Y = 2n\lambda \overline{X}_n \sim \chi^2(2n)$.

If $W = W_1 + W_2$ with W_1 and W_2 independent, $W \sim \chi^2(\nu_1+\nu_2)$, $W_2 \sim \chi^2(\nu_2)$ then $W_1 \sim \chi^2(\nu_1)$.

$t = \frac{Z}{\sqrt{W/\nu}} \sim t(\nu)$ $\qquad$ $F = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F(\nu_1, \nu_2)$

If $X_1, \ldots, X_n$ is a random sample from a Normal(μ, σ^2) distribution, then

- $\widehat{\mu} = \overline{X}_n$ and $\widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2 = \left(\frac{n-1}{n}\right) S^2$.
- $\overline{X}_n$ and S^2 are independent.
- $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.
- $t = \frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t(n-1)$

If $X_1, \ldots, X_{n_1} \overset{i.i.d.}{\sim}$ Normal(μ_1, σ_1^2) and $Y_1, \ldots, Y_{n_2} \overset{i.i.d.}{\sim}$ Normal(μ_2, σ_2^2) with X_i independent of Y_i, then

- $F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F(n_1-1, n_2-1)$.
- $T = \frac{\overline{X}-\overline{Y}-(\mu_1-\mu_2)}{S_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} \sim t(n_1+n_2-2)$ provided $\sigma_1^2 = \sigma_2^2$, where $S_p = \sqrt{\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}}$.

For $j = 1, \ldots, k$ and $i = 1, \ldots n_j$, $X_{i,j} \overset{ind}{\sim} N(\mu_j, \sigma^2)$. Let $\overline{X}_. = \sum_{j=1}^k \left(\frac{n_j}{n}\right) \overline{X}_j$.
$SSTO = SSB + SSW$, where $SSTO = \sum_{j=1}^k \sum_{i=1}^{n_j} (X_{i,j} - \overline{X}_.)^2$, $SSB = \sum_{j=1}^k n_j(\overline{X}_j - \overline{X}_.)^2$ and $SSW = \sum_{j=1}^k \sum_{i=1}^{n_j} (X_{i,j} - \overline{X}_j)^2$. $R^2 = \frac{SSB}{SSTO}$.
Under $H_0: \mu_1 = \cdots \mu_k$, $F = \frac{SSB/(k-1)}{SSW/(n-k)} = \left(\frac{n-k}{k-1}\right)\left(\frac{R^2}{1-R^2}\right) \sim F(k-1, n-k)$.

$\lambda(\mathbf{x}) = \frac{L(\widehat{\theta}_0, \mathbf{x})}{L(\widehat{\theta}, \mathbf{x})}$, $C_k = \{\mathbf{x} \in S : \lambda(\mathbf{x}) \leq k\}$, where $0 < k < 1$.
$G_n^2 = -2\ln\lambda(\mathbf{X}) \overset{d}{\to} Y \sim \chi^2(q)$, where q is the number of $=$ signs in H_0.

$\pi(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int f(x|t)\pi(t)\,dt} \propto f(x|\theta)\pi(\theta)$ $\qquad$ $E[f(x|\Theta)|\mathbf{x}] = \int f(x|\theta)\,\pi(\theta|\mathbf{x})\,d\theta$

$L(\theta, \mathbf{x}) = g[\theta, t(\mathbf{x})]\,h(\mathbf{x})$ $\qquad$ $I(\theta) = E\left(\frac{\partial}{\partial\theta}\ln f(X|\theta)\right)^2 = -E\left(\frac{\partial^2}{\partial\theta^2}\ln f(X|\theta)\right)$

If $E(T) = \theta$, $Var(T) \geq \frac{1}{n\,I(\theta)}$ $\qquad$ $\frac{\sqrt{n}(\widehat{\Theta}_n-\theta)}{\sqrt{1/I(\theta)}} \overset{d}{\to} Z \sim N(0,1)$ $\qquad$ $\frac{\sqrt{n}(\widehat{\Theta}_n-\theta)}{\sqrt{1/I(\widehat{\Theta}_n)}} \overset{d}{\to} Z \sim N(0,1)$

R Code

Distribution	**CDF** $F(x\|\theta)$	**Quantile**
Exponential(λ)	`pexp(2.7,2)`	`qexp(0.99,1)`
Gamma(α, λ)	`pgamma(1,shape=21,rate=23.35)`	`qgamma(0.025,4,23.25)`
Normal(μ, σ)	`pnorm(160,mean=100,sd=15)`	`qnorm(0.975,0,1)`
Chi-squared(ν)	`pchisq(1.7,df=4)`	`qchisq(0.95,1)`
$t(\nu)$	`pt(2.14,df=10)`	`qt(0.975,df=134)`
$F(\nu_1, \nu_2)$	`pf(3.17,df1=6,df2=114)`	`qf(0.95,6,114)`
Beta(α, β)	`pbeta(0.25,61,41)`	`qbeta(0.5,61,41)`

D.2 Standard Normal Cdf

If $Z \sim N(0, 1)$, then we can use Table D.2 to compute the cumulative distribution function (cdf) Φ for Z. For example, suppose we want to compute $\Phi(z) = P(Z < 1.03)$. The symmetry of the $N(0, 1)$ distribution about 0 implies that $\Phi(z) = 1 - \Phi(-z)$, so using Table D.2, we have that $P(Z < 1.03) = P(Z < 1.03) = 1 - P(Z < -1.03) = 1 - 0.1515 = 0.8485$.

Table D.2 Standard Normal Cdf

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

D.4 t Distribution Quantiles

Table D.4 contains some quantiles for t or Student distributions. For example, if $X \sim t(df)$, with $df = 10$ and $P = 0.98$, then $x_{0.98} = 2.359$ is the 0.98 quantile of the $t(10)$ distribution. Recall that the $t(df)$ distribution is symmetric about 0 so, for example, $x_{0.25} = -x_{0.75}$.

Table D.4 $t(df)$ Quantiles

	P									
df	0.75	0.85	0.90	0.95	0.975	0.98	0.99	0.995	0.9975	0.999
1	1.000	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3
2	0.816	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33
3	0.765	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21
4	0.741	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173
5	0.727	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.893
6	0.718	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208
7	0.711	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785
8	0.706	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501
9	0.703	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297
10	0.700	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144
11	0.697	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025
12	0.695	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930
13	0.694	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852
14	0.692	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787
15	0.691	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733
16	0.690	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686
17	0.689	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646
18	0.688	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.611
19	0.688	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579
20	0.687	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552
21	0.686	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527
22	0.686	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505
23	0.685	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485
24	0.685	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467
25	0.684	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450
26	0.684	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435
27	0.684	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421
28	0.683	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408
29	0.683	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396
30	0.683	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385
40	0.681	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307
50	0.679	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261
60	0.679	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232
80	0.678	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195
100	0.677	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174
1000	0.675	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098
∞	0.674	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.091
	50%	70%	80%	90%	95%	96%	98%	99%	99.5%	99.8%
	Confidence level									

Quantiles of the chi-squared Distribution

df	Quantile 0.005	0.025	0.05	0.95	0.975	0.995
1	0.00	0.00	0.00	3.84	5.02	7.88
2	0.01	0.05	0.10	5.99	7.38	10.60
3	0.07	0.22	0.35	7.81	9.35	12.84
4	0.21	0.48	0.71	9.49	11.14	14.86
5	0.41	0.83	1.15	11.07	12.83	16.75
6	0.68	1.24	1.64	12.59	14.45	18.55
7	0.99	1.69	2.17	14.07	16.01	20.28
8	1.34	2.18	2.73	15.51	17.53	21.95
9	1.73	2.70	3.33	16.92	19.02	23.59
10	2.16	3.25	3.94	18.31	20.48	25.19
11	2.60	3.82	4.57	19.68	21.92	26.76
12	3.07	4.40	5.23	21.03	23.34	28.30
13	3.57	5.01	5.89	22.36	24.74	29.82
14	4.07	5.63	6.57	23.68	26.12	31.32
15	4.60	6.26	7.26	25.00	27.49	32.80
16	5.14	6.91	7.96	26.30	28.85	34.27
17	5.70	7.56	8.67	27.59	30.19	35.72
18	6.26	8.23	9.39	28.87	31.53	37.16
19	6.84	8.91	10.12	30.14	32.85	38.58
20	7.43	9.59	10.85	31.41	34.17	40.00
21	8.03	10.28	11.59	32.67	35.48	41.40
22	8.64	10.98	12.34	33.92	36.78	42.80
23	9.26	11.69	13.09	35.17	38.08	44.18
24	9.89	12.40	13.85	36.42	39.36	45.56
25	10.52	13.12	14.61	37.65	40.65	46.93
26	11.16	13.84	15.38	38.89	41.92	48.29
27	11.81	14.57	16.15	40.11	43.19	49.64
28	12.46	15.31	16.93	41.34	44.46	50.99
29	13.12	16.05	17.71	42.56	45.72	52.34
30	13.79	16.79	18.49	43.77	46.98	53.67
40	20.71	24.43	26.51	55.76	59.34	66.77
50	27.99	32.36	34.76	67.50	71.42	79.49
60	35.53	40.48	43.19	79.08	83.30	91.95
80	51.17	57.15	60.39	101.88	106.63	116.32
100	67.33	74.22	77.93	124.34	129.56	140.17