Name ________________________________

Student Number ________________________________

STA 260s20 Test 2: Version B

Tutorial Section (Circle One)

TUT0101 Tues. 3-4 IB 200 Karan	TUT0102 Tues. 4-5 IB 200 Karan	TUT0103 Wed. 5-6 IB 220 Marie	TUT0104 Wed. 5-6 IB 200 Karan
TUT0105 Wed. 6-7 DV 1148 Dashvin	TUT0106 Fri. 3-4 IB 200 Michael	TUT0107 Fri. 4-5 IB 200 Michael	TUT0108 Fri. 5-6 IB 200 Marie

Question	Value	Score
1	14	
2	14	
3	10	
4	10	
5	6	
6	24	
7	22	
Total = 100 Points		

1. (14 points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = 5\theta x^{5\theta-1} I(0 < x < 1)$, where $\theta > 0$.

   (a) Find the maximum likelihood estimator of θ. Show your work. *Do not bother with the second derivative test.* Your final answer is a formula. **Circle your answer**.

   (b) Using your answer to Question 1a, give a maximum likelihood estimate for the following data: `0.903 0.857 0.769 0.297 0.999`. Show a little work. Your answer is a number. **Circle your answer**.

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2. (14 points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = 5\theta x^{5\theta-1} I(0 < x < 1)$, where $\theta > 0$. This is a repeat of the density in Question 1.

   (a) Find a method of moments estimator of θ. Show your work. Your final answer is a formula. **Circle your answer**.

   (b) Using your answer to Question 2a, give a method of moments estimate for the following data: `0.903 0.857 0.769 0.297 0.999`. This is a repeat of the data in Question 1. Show a little work. Your answer is a number. **Circle your answer**.

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3. (10 points) Let Y_1 and Y_2 be independent random variables, and let $Y = Y_1 + Y_2$. Prove that if $Y \sim \chi^2(\nu_1 + \nu_2)$ and $Y_2 \sim \chi^2(\nu_2)$, then $Y_1 \sim \chi^2(\nu_1)$. You have more room than you need.

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4. (10 points) Show that $\sum_{i=1}^{n} (X_i - \mu)^2 = \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2 + n \left(\overline{X}_n - \mu\right)^2$.

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5. (6 points) Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. Write the answers in the spaces below. No proof is required.

   (a) What is the distribution of $\overline{X}_n$? Include the parameter(s).

   (b) What is the distribution of $\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$? Include the parameter(s).

   (c) What is the distribution of $\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2$? Include the parameter(s).

   (d) What is the distribution of $\frac{X_i - \mu}{\sigma}$? Include the parameter(s).

   (e) What is the distribution of $\left(\frac{X_i - \mu}{\sigma}\right)^2$? Include the parameter(s).

   (f) What is the distribution of $\displaystyle\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$? Include the parameter(s).

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6. (24 points) Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. In this question, you are going to prove that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ by filling in the blanks. You may use the fact that $\overline{X}_n$ and S^2 are independent, without proof.

   (a) Using the results of Questions 3 and 4, what is Y in terms of the current problem? Show a little work.

   (b) What is Y_1?

   (c) What is Y_2?

   (d) What is the distribution of Y, including the parameter(s)? The symbols ν_1 and ν_2 do not appear in your answer.

   (e) Give the facts from Question 5 that prove your answer to Question 6d. Just write the letters.

   (f) What is the distribution of Y_2, including the parameter(s)? The symbol ν_2 does not appear in your answer.

   (g) Give the facts from Question 5 that prove your answer to Question 6f. Just write the letters.

   (h) How do you know that Y_1 and Y_2 are independent?

   (i) Therefore, the distribution of Y_1 is ... What? Include the parameter(s).

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7. (22 points) Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. We want to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$.

   (a) Write down the formula for a good test statistic. No explanation is needed.

   (b) What is the distribution of the test statistic under the null hypothesis? Include the parameter(s).

   (c) What is the decision rule? That is, when is the null hypothesis to be rejected? Your answer uses the quantiles of the distribution you specified in Question 7b.

   (d) A sample of size $n = 51$ yields a sample mean of $\overline{x}_n = 50.52$ and a sample variance of $S^2 = 61.45$. We want to test the null hypothesis $\sigma^2 = 100$ at the $\alpha = 0.05$ significance level.

      i. What are the critical values? Your answer is a pair of numbers.

      ii. What is the computed value of the test statistic? Show a little work. Your answer is a number. **Circle your final answer.**

      iii. Do you reject the null hypothesis? Answer Yes or No.

      iv. What do you conclude from the test? Circle one of the conclusions below.

      $\sigma^2 > 100$.

      $\sigma^2 < 100$.

      $\sigma^2 = 100$.

Total Marks = 100 points