Name Jerry

Student Number

STA 260s20 Test 2: Version B

Tutorial Section (Circle One)

TUT0101 Tues. 3-4 IB 200 Karan	TUT0102 Tues. 4-5 IB 200 Karan	TUT0103 Wed. 5-6 IB 220 Marie	TUT0104 Wed. 5-6 IB 200 Karan
TUT0105 Wed. 6-7 DV 1148 Dashvin	TUT0106 Fri. 3-4 IB 200 Michael	TUT0107 Fri. 4-5 IB 200 Michael	TUT0108 Fri. 5-6 IB 200 Marie

Question	Value	Score
1	14	
2	14	
3	10	
4	10	
5	6	
6	24	
7	22	
Total = 100 Points		

1. (14 points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = 5\theta x^{5\theta-1} I(0 < x < 1)$, where $\theta > 0$.

   (a) Find the maximum likelihood estimator of θ. Show your work. *Do not bother with the second derivative test.* Your final answer is a formula. **Circle your answer.**

$$\ell(\theta) = \ln \prod_{i=1}^{n} 5\theta x_i^{5\theta-1} = \ln\left[5^n \theta^n \left(\prod_{i=1}^{n} x_i\right)^{5\theta-1}\right]$$

$$= n\ln 5 + n\ln\theta + (5\theta - 1)\sum_{i=1}^{n} \ln x_i$$

$$\ell'(\theta) = \frac{n}{\theta} + 5\sum_{i=1}^{n} \ln x_i \overset{\text{set}}{=} 0$$

$$\Longrightarrow \frac{n}{\theta} = -5\sum_{i=1}^{n} \ln x_i$$

$$\Longrightarrow \hat{\theta} = \frac{n}{-5\sum_{i=1}^{n} \ln x_i}$$

   (b) Using your answer to Question 1a, give a maximum likelihood estimate for the following data: `0.903 0.857 0.769 0.297 0.999`. Show a little work. Your answer is a number. **Circle your answer.**

$$\hat{\theta} = \frac{\cancel{5}}{-\cancel{5}\sum_{i=1}^{n} \ln x_i} = \frac{-1}{\sum_{i=1}^{n} \ln x_i} = \frac{-1}{-1.734}$$

$$= 0.577$$

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2. (14 points) Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = 5\theta x^{5\theta-1} I(0 < x < 1)$, where $\theta > 0$. This is a repeat of the density in Question 1.

   (a) Find a method of moments estimator of θ. Show your work. Your final answer is a formula. **Circle your answer.**

$$E(X) = \int_0^1 x \, 5\theta x^{5\theta-1} \, dx = 5\theta \int_0^1 x^{5\theta} \, dx$$

$$= 5\theta \left. \frac{x^{5\theta+1}}{5\theta+1} \right|_0^1 = \frac{5\theta}{5\theta+1}, \text{ so set}$$

$$\bar{x} = \frac{5\hat{\theta}}{5\hat{\theta}+1} \iff 5\hat{\theta}\bar{x} + \bar{x} = 5\hat{\theta}$$

$$\Rightarrow \bar{x} = 5\hat{\theta} - 5\hat{\theta}\bar{x} = 5\hat{\theta}(1-\bar{x})$$

$$\Rightarrow \hat{\theta} = \frac{\bar{x}}{5(1-\bar{x})}$$

   (b) Using your answer to Question 2a, give a method of moments estimate for the following data: `0.903 0.857 0.769 0.297 0.999`. This is a repeat of the data in Question 1. Show a little work. Your answer is a number. **Circle your answer.**

$$\bar{x} = 0.765, \text{ and } \hat{\theta} = \frac{0.765}{5(1-0.765)}$$

$$= \frac{0.765}{5(0.235)} = \frac{0.765}{1.175} = 0.651$$

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3. (10 points) Let Y_1 and Y_2 be independent random variables, and let $Y = Y_1 + Y_2$. Prove that if $Y \sim \chi^2(\nu_1 + \nu_2)$ and $Y_2 \sim \chi^2(\nu_2)$, then $Y_1 \sim \chi^2(\nu_1)$. You have more room than you need.

$$M_Y(t) \overset{\text{ind}}{=} M_{Y_1}(t)\, M_{Y_2}(t)$$

$$\Rightarrow (1-2t)^{-\frac{\nu_1+\nu_2}{2}} = M_{Y_1}(t)\,(1-2t)^{-\nu_2/2}$$

$$\Rightarrow (1-2t)^{-\nu_1/2}\,\cancel{(1-2t)^{-\nu_2/2}} = M_{Y_1}(t)\,\cancel{(1-2t)^{-\nu_2/2}}$$

$$\Rightarrow M_{Y_1}(t) = (1-2t)^{-\nu_1/2}$$

MGF of $\chi^2(\nu_1)$

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4. (10 points) Show that $\sum_{i=1}^{n}(X_i-\mu)^2=\sum_{i=1}^{n}\left(X_i-\overline{X}_n\right)^2+n\left(\overline{X}_n-\mu\right)^2$.

$$\sum_{i=1}^{n}(X_i-\mu)^2=\sum_{i=1}^{n}(X_i-\overline{X}_n+\overline{X}_n-\mu)^2$$

$$=\sum_{i=1}^{n}\left[(X_i-\overline{X}_n)^2+2(X_i-\overline{X}_n)(\overline{X}_n-\mu)+(\overline{X}_n-\mu)^2\right]$$

$$=\sum_{i=1}^{n}(X_i-\overline{X}_n)^2+2(\overline{X}_n-\mu)\underbrace{\sum_{i=1}^{n}(X_i-\overline{X}_n)}_{=0}+n(\overline{X}_n-\mu)^2$$

$$=\sum_{i=1}^{n}(X_i-\overline{X}_n)^2+n(\overline{X}_n-\mu)^2$$

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5. (6 points) Let $X_1, \dots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. Write the answers in the spaces below. No proof is required.

   (a) What is the distribution of $\overline{X}_n$? Include the parameter(s).

   $$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

   (b) What is the distribution of $\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$? Include the parameter(s).

   $$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

   (c) What is the distribution of $\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}\right)^2$? Include the parameter(s).

   $$\chi^2(1)$$

   (d) What is the distribution of $\frac{X_i - \mu}{\sigma}$? Include the parameter(s).

   $$N(0,1)$$

   (e) What is the distribution of $\left(\frac{X_i - \mu}{\sigma}\right)^2$? Include the parameter(s).

   $$\chi^2(1)$$

   (f) What is the distribution of $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2$? Include the parameter(s).

   $$\chi^2(n)$$

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6. (24 points) Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. In this question, you are going to prove that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ by filling in the blanks. You may use the fact that $\overline{X}_n$ and S^2 are independent, without proof.

   (a) Using the results of Questions 3 and 4, what is Y in terms of the current problem? Show a little work.

   $$Y = \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

   (b) What is Y_1?

   $$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X}_n)^2 = \frac{(n-1)S^2}{\sigma^2}$$

   (c) What is Y_2?

   $$\frac{1}{\sigma^2}n(\overline{X}_n-\mu)^2 = \left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$$

   (d) What is the distribution of Y, including the parameter(s)? The symbols ν_1 and ν_2 do not appear in your answer.

   $\chi^2(n)$

   (e) Give the facts from Question 5 that prove your answer to Question 6d. Just write the letters.

   d, e, f

   (f) What is the distribution of Y_2, including the parameter(s)? The symbol ν_2 does not appear in your answer.

   $\chi^2(1)$

   (g) Give the facts from Question 5 that prove your answer to Question 6f. Just write the letters.

   a, b, c

   (h) How do you know that Y_1 and Y_2 are independent?

   Because $\overline{X}_n$ and S^2 are independent

   (i) Therefore, the distribution of Y_1 is ... What? Include the parameter(s).

   $\chi^2(n-1)$

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7. (22 points) Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution. We want to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$.

(a) Write down the formula for a good test statistic. No explanation is needed.

$$Y = \frac{(n-1)S^2}{\sigma_0^2}$$

(b) What is the distribution of the test statistic under the null hypothesis? Include the parameter(s). $\chi^2(n-1)$

(c) What is the decision rule? That is, when is the null hypothesis to be rejected? Your answer uses the quantiles of the distribution you specified in Question 7b.

Reject H_0 if $Y \leq \chi^2_{\alpha/2}(n-1)$ or

$Y \geq \chi^2_{1-\alpha/2}(n-1)$

(d) A sample of size $n = 51$ yields a sample mean of $\overline{x}_n = 50.52$ and a sample variance of $S^2 = 61.45$. We want to test the null hypothesis $\sigma^2 = 100$ at the $\alpha = 0.05$ significance level.

i. What are the critical values? Your answer is a pair of numbers.

32.36 , 71.42

ii. What is the computed value of the test statistic? Show a little work. Your answer is a number. **Circle your final answer.**

$$Y = \frac{(5\cancel{1}-1)\,61.45}{100} = 30.725$$

iii. Do you reject the null hypothesis? Answer Yes or No. Yes

iv. What do you conclude from the test? Circle one of the conclusions below.

$\sigma^2 > 100.$

$\sigma^2 < 100.$

$\sigma^2 = 100.$

Total Marks = 100 points