Assignment 9

① (a) $p(\underset{\sim}{x} \mid \lambda) = \prod_{i=1}^{n} \dfrac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \dfrac{e^{-n\lambda}\lambda^{n\bar{x}}}{\prod_{i=1}^{n} x_i!}$

$$\pi(\theta \mid \underset{\sim}{x}) \propto \frac{e^{-n\lambda}\lambda^{n\bar{x}}}{\prod_{i=1}^{n} x_i!} \frac{\beta^{\alpha}}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} I(\lambda \geq 0)$$

$$\propto e^{-(\beta+n)\lambda} \lambda^{\alpha+n\bar{x}-1} I(\lambda \geq 0)$$

$$\propto \frac{(\beta+n)^{\alpha+n\bar{x}}}{\Gamma(\alpha+n\bar{x})} e^{-(\beta+n)\lambda} \lambda^{\alpha+n\bar{x}-1} I(\lambda \geq 0)$$

Gamma (α', β'), $\alpha' = \alpha + n\bar{x}$, $\beta' = \beta + n$

(b) $$p(x \mid \underset{\sim}{x}) = \int_0^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} \, I(x=0,1,\ldots) \, \frac{(\beta+n)^{\alpha+n\bar{x}}}{\Gamma(\alpha+n\bar{x})} e^{-(\beta+n)\lambda} \lambda^{\alpha+n\bar{x}-1} \, d\lambda$$

$$= \frac{(\beta+n)^{\alpha+n\bar{x}}}{x!\,\Gamma(\alpha+n\bar{x})} \frac{\Gamma(\alpha+n\bar{x}+x)}{\beta^{\alpha+n\bar{x}+x}} I(x=0,1,\ldots$$

$$\underbrace{\int_0^{\infty} \frac{(\beta+n+1)^{\alpha+n\bar{x}+x}}{\Gamma(\alpha+n\bar{x}+x)} e^{-(\beta+n+1)\lambda} \lambda^{(\alpha+n\bar{x}+x)-1} \, d\lambda}_{=1}$$

$$= \frac{\Gamma(\alpha+n\bar{x}+x)}{x!\,\Gamma(\alpha+n\bar{x})\,\beta^{x}} I(x=0,1,\ldots)$$

I don't recognize it either.

(2) (a) $p(\underset{\sim}{x} \mid \theta) = \prod_{i=1}^{n} \binom{4}{x_i} \theta^{x_i} (1-\theta)^{4-x_i}$

$= \prod_{i=1}^{n} \binom{4}{x_i} \theta^{\Sigma x_i} (1-\theta)^{4n - \Sigma x_i}$

$\pi(\theta \mid \underset{\sim}{x}) \propto p(\underset{\sim}{x} \mid \theta) \pi(\theta)$

$= \left(\prod_{i=1}^{n} \binom{4}{x_i} \right) \theta^{n\bar{x}} (1-\theta)^{4n - n\bar{x}} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} I(0 \le \theta \le 1)$

$\propto \theta^{\alpha + n\bar{x} - 1} (1-\theta)^{(\beta + 4n - n\bar{x}) - 1} I(0 \le \theta \le 1)$

$\propto \dfrac{\Gamma(\alpha + \cancel{n\bar{x}} + \beta + 4n - \cancel{n\bar{x}})}{\Gamma(\alpha + n\bar{x}) \Gamma(\beta + 4n - n\bar{x})} \theta^{\alpha + n\bar{x} - 1} (1-\theta)^{\beta + 4n - n\bar{x} - 1} I(0 \le \theta \le 1)$

Beta (α', β')

b) For $n = 20$, $\alpha = \beta = 1$, $\bar{x}_n = 2.3$,

(i) $\dfrac{\alpha'}{\alpha' + \beta'} = \dfrac{1 + (20)(2.3)}{47 + 1 + (4)(20) - (20)(2.3)} = \dfrac{47}{47 + 35}$

$= \dfrac{47}{82} \approx 0.57$

(2b ii) Finding mode of a general Beta (α, β)

$$\frac{d}{dx} \ln\left[x^{\alpha-1}(1-x)^{\beta-1}\right] = \frac{d}{dx}\left((\alpha-1)\ln x + (\beta-1)\ln(1-x)\right)$$

$$= \frac{\alpha-1}{x} - \frac{\beta-1}{1-x} \overset{\text{set}}{=} 0 \iff x(\beta-1) = \alpha-1 - x(\alpha-1)$$

$$\implies \alpha-1 = x(\alpha+\beta-2) \implies x = \frac{\alpha-1}{\alpha+\beta-2},$$

So posterior mode is $\dfrac{\alpha'-1}{\alpha'+\beta'-2}$

$$= \frac{\alpha+n\bar{x}-1}{\alpha+\cancel{n\bar{x}}+\beta+4n-\cancel{n\bar{x}}} = \frac{\alpha+n\bar{x}-1}{\alpha+\beta+4n}$$

For these data, $= \dfrac{1+(20)(2.3)-1}{1+1+(4)(20)} = \dfrac{46}{81}$

$$\approx 0.568$$

(iii) A. $P(\Theta = \frac{1}{2} \mid \underset{\sim}{x}) = 0$

B. $P(\Theta < \frac{1}{2} \mid \underset{\sim}{x}) = \text{pbeta}(1/2, 47, 35) = 0.09$

$P(\Theta > \frac{1}{2} \mid x = 1 - \uparrow \approx 0.91$

About a 90% chance the coin is biased with $P(H) > \frac{1}{2}$.

(2 b iv) $qbeta(0.025, 47, 35) = 0.465$

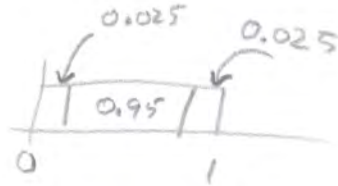
$qbeta(0.975, 47, 35) = 0.678$

so $P(0.465 < \theta < 0.678) = 0.95$

Lower limit ... upper limit

v) Sure: $(0.025, 0.975)$

why not?



(3) (a) $\pi(\lambda \mid \underset{\sim}{x}) \propto \lambda^n e^{-\lambda n \bar{x}} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\lambda} \lambda^{\alpha-1} I(\lambda \geq 0)$

$\propto e^{-(\beta + n\bar{x})\lambda} \lambda^{\alpha+n-1} I(\lambda \geq 0)$ Gamma(α', β')

$\alpha' = (\alpha + n), \quad \beta' = \beta + n\bar{x}$

(b) For a general Gamma(α, β)

$\frac{d}{dx} \ln f(x \mid \alpha, \beta) = \frac{d}{dx} \ln \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}$

$= \frac{d}{dx} \left(\ln \frac{\beta^\alpha}{\Gamma(\alpha)} - \beta x + (\alpha - 1) \ln x \right)$

$= -\beta + \frac{(\alpha-1)}{x} \overset{set}{=} 0 \Rightarrow x\beta = \alpha - 1$

$\Rightarrow x = \frac{\alpha - 1}{\beta}$. 2nd derivative $= \frac{d}{dx} (\alpha-1) x^{-1}$

$= (\alpha-1)(-1) x^{-2}$

$= \frac{(1-\alpha)}{x^2}$ concave down for $\alpha > 1$

So the posterior mode is $\frac{\alpha' - 1}{\beta'}$

$= \frac{\alpha + n - 1}{\beta + n\bar{x}}$

(3c)

Formula sheet

$$E(\lambda \mid \underset{\sim}{X}) \overset{\downarrow}{=} \frac{\alpha'}{\beta'} = \frac{\alpha+n}{\beta+n\bar{X}_n} = \frac{\frac{1}{n}(\alpha+n)}{\frac{1}{n}(\beta+n\bar{x})}$$

$$= \frac{\frac{\alpha}{n}+1}{\frac{\beta}{n}+\bar{X}_n} \xrightarrow{P} \frac{1}{E(X_i)} = \frac{1}{1/\lambda_0} = \lambda_0$$

Yes, consistent by LLN & continuous mapping.

(d) $$Var(\lambda \mid \underset{\sim}{X}) = \frac{\alpha'}{\beta'^2} = \frac{\alpha+n}{(\beta+n\bar{X}_n)^2}$$

$$= \frac{\frac{1}{n^2}(\alpha+n)}{\frac{1}{n^2}(\beta+n\bar{X}_n)^2}$$

$$= \frac{\alpha/n^2 + 1/n}{\left(\beta/n + \bar{X}_n\right)^2} \xrightarrow{P} \frac{0}{(0+1/\lambda_0)^2} = 0$$

By continuous mapping & LLN.

(e) $$\frac{\frac{1}{n}(\alpha+n-1)}{\frac{1}{n}(\beta+n\bar{X}_n)} = \frac{\alpha/n + 1 - 1/n}{\beta/n + \bar{X}_n} \xrightarrow{P} \frac{1}{1/\lambda_0}$$

$= \lambda_0$ by continuous mapping and LLN.

Yes, consistent

(4) (a) $L(\tau, \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{\tau^{1/2}}{\sqrt{2\pi}} e^{-\frac{\tau}{2}(x_i-\mu)^2}$

$= \frac{\tau^{n/2}}{(2\pi)^{n/2}} e^{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2}$, and

$\pi(\tau \mid \underset{\sim}{x}) \propto \tau^{n/2} e^{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2} \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta\tau} \tau^{\alpha-1} I(\tau \geq 0)$

$\propto e^{-(\beta+\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2)\tau} \tau^{\alpha+n/2-1} I(\tau \geq 0)$

Gamma(α', β') with $\alpha' = \alpha + \frac{n}{2}$

$\beta' = \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2$

(b) $\pi(\mu \mid \underset{\sim}{x}) \propto e^{-\frac{\tau}{2}\sum_{i=1}^{n}(x_i-\mu)^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2}$

$\propto \text{Exp} -\frac{1}{2}\left[\left(\tau \sum_{i=1}^{n}(x_i^2 - 2x_i\mu + \mu^2)\right) + \mu^2\right]$

$= \text{Exp} -\frac{1}{2}\left(\underbrace{\tau \sum_{i=1}^{n} x_i^2}_{} - 2\tau n \bar{x}\mu + \tau n \mu^2 + \mu^2\right)$

$\propto \text{Exp} -\frac{1}{2}\left[(\tau n+1)\mu^2 - 2\tau n \bar{x}\mu\right]$

$= \text{Exp} -\frac{(\tau n+1)}{2}\left(\mu^2 - 2\mu\left(\frac{\tau n \bar{x}}{\tau n+1}\right)\right)$

(4b continued)

$$\propto \operatorname{Exp} - \frac{(\tau n+1)}{2}\left(\mu^2 - 2\mu\left(\frac{\tau n \bar{x}}{\tau n+1}\right) + \left(\frac{\tau n \bar{x}}{\tau n+1}\right)^2\right)$$

$$\propto \frac{(\tau n+1)^{1/2}}{\sqrt{2\pi}} \operatorname{Exp} - \frac{\tau n+1}{2}\left(\mu - \frac{\tau n \bar{x}_n}{\tau n+1}\right)^2$$

$$\text{Normal}\left(\frac{\tau n \bar{x}_n}{\tau n+1},\ \tau n+1\right)$$

(5) Denote the posterior for $\pi_j(\theta)$ by $\pi_j(\theta|\underset{\sim}{x}) \propto f(\underset{\sim}{x}|\theta)\pi_j(\theta)$. If the prior is conjugate, $\pi_j(\theta)$ and $\pi_j(\theta|\underset{\sim}{x})$ are in the same family.

$$\pi(\theta|\underset{\sim}{x}) \propto f(\underset{\sim}{x}|\theta)\,\pi(\theta) = f(\underset{\sim}{x}|\theta)\sum_{j=1}^{k} a_j \pi_j(\theta)$$

$$= \sum_{j=1}^{k} a_j\, f(\underset{\sim}{x}|\theta)\,\pi_j(\theta)$$

$$= \sum_{j=1}^{k}\left(a_j \int f(\underset{\sim}{x}|t)\pi_j(t)\,dt\right)\frac{f(\underset{\sim}{x}|\theta)\pi_j(\theta)}{\int f(\underset{\sim}{x}|t)\pi_j(t)\,dt}$$

$$= \sum_{j=1}^{k} a_j \int f(\underset{\sim}{x}|t)\pi_j(t)\,dt\;\pi_j(\theta|\underset{\sim}{x}) = \sum_{j=1}^{k} a_j P_j(\underset{\sim}{x})\,\pi_j(\theta|\underset{\sim}{x})$$

Where $P_j(\underset{\sim}{x})$ is the prior predictive joint density for prior $\pi_j(\theta)$.

This is a linear combination of posteriors, but not a mixture because the weights don't add to one.

However, consider the denominator of the overall posterior $\pi(\theta|\underset{\sim}{x})$, the prior predictive joint density:

$$p(\underset{\sim}{x}) = \int f(\underset{\sim}{x}|t)\,\pi(t)\,dt = \int f(\underset{\sim}{x}|t)\sum_{i=1}^{k} a_i\,\pi_i(t)\,dt$$

$$= \sum_{i=1}^{k} a_i \int f(x|t)\,\pi_i(t)\,dt = \sum_{i=1}^{k} a_i P_i(\underset{\sim}{x})$$

(5 continued)

So, the complete posterior is

$$\pi(\theta \mid \underset{\sim}{x}) = \frac{\sum_{j=1}^{k} a_j P_j(\underset{\sim}{x})\, \pi_j(\theta \mid \underset{\sim}{x})}{\sum_{i=1}^{k} a_i P_i(\underset{\sim}{x})}$$

$$= \sum_{j=1}^{k} \underbrace{\left(\frac{a_j P_j(\underset{\sim}{x})}{\sum_{i=1}^{k} a_i P_i(\underset{\sim}{x})} \right)} \pi_j(\theta \mid \underset{\sim}{x})$$

The new mixing weights add to one.