STA 260s20 Assignment Eight: Hypothesis testing Part 2 – Likelihood Ratio Tests[1]

The following homework problems are not to be handed in. They are preparation for the final exam. **Please try each question before looking at the solution**. Use the formula sheet.

1. For random sampling from two or more normal distributions with the same variance, the formula sheet has the t-test of $H_0 : \mu_1 = \mu_2$ and the F-test of $H_0 : \mu_1 = \cdots \mu_k$. Show that if $k = 2$, $F = T^2$.

2. Let $X_1, \ldots, X_{n_1} \stackrel{i.i.d.}{\sim}$ Normal(μ_1, σ^2) and $Y_1, \ldots, Y_{n_2} \stackrel{i.i.d.}{\sim}$ Normal(μ_2, σ^2) with all the X_i independent of all the Y_i. Show that the usual two-sample $t-$test of $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ is an exact likelihood ratio test.

3. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta) = \frac{1}{2\theta^3} x^2 e^{-x/\theta} \, I(x > 0)$, where the parameter $\theta > 0$.

   (a) What is the distribution of the MLE $\widehat{\Theta}$? Show your work; use moment-generating functions.

   (b) Find an exact size α chi-squared likelihood ratio test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Do not forget to show that the test is size α.

4. Let $X_1, \ldots, X_n$ be a random sample from a geometric distribution with parameter θ. We seek to test $H_0 : \theta = \frac{1}{2}$ versus $H_0 : \theta \neq \frac{1}{2}$. A sample of size $n = 50$ yields $\overline{x} = 1.56$ and $s^2 = 4.17$.

   (a) Write a formula for the large-sample likelihood ratio test statistic. Simplify!

   (b) Calculate G^2_n. What is the critical value at $\alpha = 0.05$?

   (c) Do you reject H_0? Answer Yes or No.

   (d) What do you conclude? Choose one: $\theta < \frac{1}{2}$ $\quad \theta = \frac{1}{2}$ $\quad \theta > \frac{1}{2}$

5. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with unknown expected value and unknown variance.

   (a) The null hypothesis is that the distribution is standard normal. Obtain and simplify a formula for the large-sample likelihood ratio test statistic G^2_n.

   (b) A sample of size $n = 200$ yields $\bar{x}_n = 0.062$ and $\widehat{\sigma}^2_n = 1.353$.

      i. Calculate G^2_n.

      ii. What are the degrees of freedom?

      iii. What is the critical value at significance level $\alpha = 0.05$?

      iv. Do you reject H_0? Answer Yes or No.

      v. Is it possible to draw a directional conclusion here?

[1] Copyright information is at the end of the last page.

6. Dead pixels are a big problem in manufacturing computer and cell phone screens. The physics of the manufacturing process dictates that dead pixels happen according to a spatial Poisson process, so that the numbers of dead pixels in cell phone screens are independent Poisson random variables with parameter λ, the expected number of dead pixels. Naturally, λ depends on details of how the screens are manufactured.

   In an effort to reduce the expected number of dead pixels, six assembly lines were set up, each with a different version of the manufacturing process. A random sample of 50 phones was taken from each assembly line and sent to the lab for testing. Mysteriously, three phones from one assembly line disappeared in transit, and 15 phones from another assembly line disappeared. Sample sizes and sample mean numbers of dead pixels appear in the table below.

```
                    Manufacturing Process
              1        2        3      4      5         6
           ----------------------------------------
      ybar 10.68   9.87234  9.56   8.52 10.48571  9.98
       n    50       47      50     50     35       50
           ----------------------------------------
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   (a) What is the parameter space Ω?
   (b) We want to know whether the expected number of dead pixels is different for the six manufacturing processes. What is the null hypothesis, in symbols? Use Greek letters.
   (c) What is the alternative hypothesis?
   (d) What is Ω_0?
   (e) What is Ω_1?
   (f) What is $\widehat{\theta}$? The answer is numerical. Note that it is a point in the parameter space.
   (g) What is $\widehat{\theta}_0$? The answer is numerical. Like $\widehat{\theta}$, it is a point in the parameter space.
   (h) Give a formula for G^2. Keep simplifying!
   (i) Calculate G^2 for the data above.
   (j) What are the degrees of freedom?
   (k) What is the critical value at $\alpha = 0.05$?
   (l) Do you reject H_0? Answer Yes or No.
   (m) Is a directional conclusion possible here? Answer Yes or No. (In practice we would follow up with tests comparing all $\binom{7}{2}$ pairs of means.)
