STA 260 Assignment 6

① (a) $Y \sim N(a\mu + b, a^2\sigma^2)$

(b) $Z \sim N(0,1)$

(c) $Y \sim \chi^2(1)$

(d) $Y \sim N(n\mu, n\sigma^2)$

(e) $Y \sim N(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2)$

(f) $\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

(g) $Z \sim N(0,1)$. No. It's exact.

(h) $\chi^2(n-1)$

(i) $t(n-1$

(j) $Y \sim \chi^2(\sum_{i=1}^{n} \nu_i)$

② (a) random sample (b) Statistic (c) Unbiased

(d) Consistent (e) Moments (f) $\frac{1}{n}\sum_{i=1}^{n} X_i^4$

(g) Parameter space (h) Sample space (i) Null hypothesis

(j) Alternative hypothesis (k) Null hypothesis

(l) Critical region (m) Test (n) Critical value

(o) Type II error (p) Type I error

(q) significance level, α (r) Significance level

(s) significance level (t) power

③ 6.3.1

(a) $Z = \dfrac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_0}$ (b) $\sim N(0,1)$, exact

(c) i) $\Omega_0 = \{\mu_0\}$ ii) $\Omega_1 = \{\mu \in \mathbb{R} : \mu \neq \mu_0\}$

iii) $\pm z_{1-\frac{\alpha}{2}}$ iv) ± 1.96 v) ± 2.576

vi) Reject H_0 when $|Z| \geq z_{1-\alpha/2}$

vii) $\bar{x}_n = 4.88$, $Z = \dfrac{\sqrt{10}(4.88-5)}{\sqrt{0.5}} = -0.54$

viii) No, don't reject

ix)

$p = 2\,\Phi(-0.54) = 2 * 0.2946$

$= 0.5892$

x) $\mu = 5$

xi) $\bar{x} \pm z_{1-\frac{\alpha}{2}} \frac{\sigma_0}{\sqrt{n}}$ (See formula sheet) $= 4.88 \pm 1.96 \frac{\sqrt{0.5}}{\sqrt{10}}$

$= 4.88 \pm 0.44 = (4.44, 5.32)$

(d) i) $\Omega_0 = (-\infty\ \mu_0]$ ii) $\Omega_1 = (\mu_0, \infty)$

iii) $z_{1-\alpha}$ iv) $z_{0.95} = 1.645$ v) $z_{0.99} = 2.326$

vi) Reject H_0 when $Z \geq z_{1-\alpha}$ vii) No

viii)

$p = 1 - \Phi(-0.54)$

$= 1 - 0.2946 = 0.7054$

ix) Conclude $\mu \leq 5$

(3 & x.) Want $\max_{\mu \le \mu_0} P_\mu(Z > z_{1-\alpha})$

$$P_\mu(Z \ge z_{1-\alpha}) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_0} \ge z_{1-\alpha}\right)$$

$$= P_\mu\left(\bar{X}_n - \mu_0 \ge z_{1-\alpha}\frac{\sigma_0}{\sqrt{n}}\right) = P_\mu\left(\bar{X}_n \ge \mu_0 + z_{1-\alpha}\frac{\sigma_0}{\sqrt{n}}\right)$$

$$= P_\mu\left(\bar{X}_n - \mu \ge \mu_0 - \mu + z_{1-\alpha}\frac{\sigma_0}{\sqrt{n}}\right)$$

$$= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma_0} \ge \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + \frac{\cancel{\sqrt{n}}\, z_{1-\alpha}\frac{\cancel{\sigma_0}}{\cancel{\sqrt{n}}}}{\cancel{\sigma_0}}\right)$$

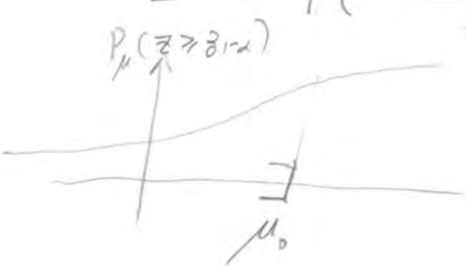
$$= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma_0} \ge \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_{1-\frac{\alpha}{2}}\right)$$

$$= 1 - \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_{1-\frac{\alpha}{2}}\right)$$ where $\Phi(z)$ is the cdf of the standard normal (*)

To maximize over $\Omega_0 = \{\mu : \mu \le \mu_0\}$,

$$\frac{d}{d\mu} P_\mu(Z \ge z_{1-\alpha}) = \frac{d}{d\mu}\left[1 - \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_{1-\alpha}\right)\right]$$

$$= -f\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_{1-\alpha}\right)\cdot\left(\frac{-\sqrt{n}}{\sigma_0}\right) > 0$$ <u>increasing</u>



$\underset{\mu \in \Omega_0}{\text{Maximum}}$ is at $\mu = \mu_0$ ✓

3d xi) The hard work has already been done. When $\mu > \mu_0$, Expression (*) on the preceding page is the power

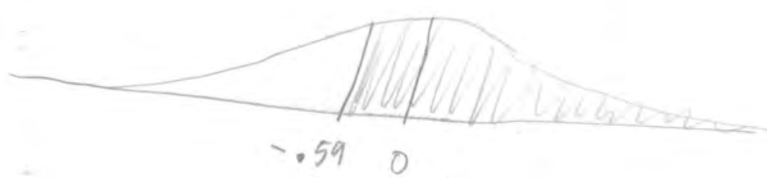
$$\text{Power}(\mu) = P_\mu(Z > z_{1-\alpha})$$

$$= 1 - \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_{1-\alpha}\right)$$

(e) Power when $\mu = 5.5$ is

$$1 - \Phi\left(\frac{\sqrt{10}(5 - 5.5)}{\sqrt{0.5}} + 1.645\right) = 1 - \Phi(-2.236 + 1.645)$$

$$= 1 - \Phi(-0.59)$$



$$= 1 - 0.2776 = 0.7224$$

Power when $\mu = 6$ is

$$1 - \Phi\left(\frac{\sqrt{10}(5 - 6)}{\sqrt{0.5}} + 1.645\right)$$

$$= 1 - \Phi(-2.83) = 1 - 0.0023$$

$$= 0.9977$$

The more wrong H_0 is, the greater the probability that it will be rejected.

(3f) $H_0: \mu \geq \mu_0$ vs $H_1: \mu < \mu_0$

(i) $\Omega_0 = \{\mu : \mu \geq \mu_0\}$ (ii) $\Omega_1 = \{\mu : \mu < \mu_0\}$

(iii) $-z_{1-\alpha} = z_\alpha$

(iv) $z_{0.05} = -1.645$ (v) $z_{0.01} = -2.576$

(vi) Reject H_0 when $Z \leq z_\alpha$

(vii) $$P_\mu(Z \leq z_\alpha) = P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu_0)}{\sigma_0} \leq z_\alpha\right)$$

$$= P_\mu\left(\bar{X}_n - \mu_0 \leq z_\alpha \frac{\sigma_0}{\sqrt{n}}\right) = P_\mu\left(\bar{X}_n \leq \mu_0 + z_\alpha \frac{\sigma_0}{\sqrt{n}}\right)$$

$$= P_\mu\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma_0} \leq \frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_\alpha\right)$$

$$= \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_\alpha\right) \text{ and}$$

$$\frac{d}{d\mu} \Phi\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_\alpha\right) = f\left(\frac{\sqrt{n}(\mu_0 - \mu)}{\sigma_0} + z_\alpha\right) \cdot \left(\frac{-\sqrt{n}}{\sigma_0}\right)$$

< 0, decreasing

$\max_{\mu \in \Omega_0} P_\mu(Z < z_\alpha)$ occurs at $\mu = \mu_0$.

(3f viii) $z = -0.54 > -1.645$ No

(ix)

$\Phi(-0.54) = 0.2946$

(x) $\mu \geq 5$

(4) 6.3.2

(a) $t = \dfrac{\sqrt{n}(\bar{X}_n - \mu_0)}{s}$ (b) $\overset{H_0}{\sim} t(n-1)$

(c) (i) $\Omega_0 = \{(\mu, \sigma^2) \in \mathbb{R}^2 : \mu = \mu_0, \sigma^2 > 0$

(ii) $\Omega_1 = \{(\mu, \sigma^2) \in \mathbb{R}^2 : \mu \neq \mu_0, \sigma^2 > 0\}$

(iii) $\pm t_{1-\alpha/2}(n-1)$

(iv) For $\alpha = 0.05$, ± 2.262

(v) For $\alpha = 0.01$, ± 3.25

(vi) Reject H_0 when $|t| \geq t_{1-\alpha/2}(n-1)$

(vii) $t = \dfrac{\sqrt{10}(4.88 - 5)}{0.696} = -0.545$

(viii) No (ix) Conclude $\mu = 5$

(x) Formula sheet has $\bar{x} \pm t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}$

$= 4.88 \pm 2.262 \dfrac{0.696}{\sqrt{10}} = 4.88 \pm 0.50$

$= (4.38, 5.38)$

(5) 6.3.4 $\bar{x} = 65.75$, $s^2 = 177.5833$, $n=4$

(a) $T = \dfrac{\sqrt{n}\,(\bar{X}_n - \mu_0)}{S}$ (b) $T \sim t(n-1)$ under H_0

(c) (i) $\Omega_0 = \{(\mu, \sigma^2) \in \mathbb{R}^2 : \mu = \mu_0, \sigma^2 > 0\}$

(ii) $\Omega_1 = \{(\mu, \sigma^2) \in \mathbb{R}^2 : \mu \neq \mu_0, \sigma^2 > 0\}$

(iii) $\pm t_{1-\alpha/2}(n-1)$

(iv) $\pm t_{0.975}(3) = \pm 3.182$ (v) $\pm t_{0.995}(3) = \pm 5.841$

(vi) Reject H_0 when $|T| > t_{1-\alpha/2}(n-1)$

(vii) $t = \dfrac{\sqrt{4}\,(65.75 - 60)}{\sqrt{177.58}} = 0.863$ (viii) No

(ix) $\mu = 60$

(x) $\bar{x} \pm t_{1-\alpha/2} \dfrac{s}{\sqrt{n}} = 65.75 \pm 3.182 \dfrac{13.33}{2}$

$= 65.75 \pm 21.21 = (44.54, 86.96)$

(6) 6.3.5

$$Z = \frac{\sqrt{n}\,(\bar{X}_n - \mu_0)}{\sigma_0} = \frac{\sqrt{1}\,(52-60)}{\sqrt{5}} = -3.58$$

$p = 2 * \Phi(-3.58)$ It's not in the table, but answer $p = 0.00034$ is right

95% CI is $\bar{x} \pm z_{1-\frac{\alpha}{2}} \dfrac{\sigma_0}{\sqrt{n}} = 52 \pm 1.96 \dfrac{\sqrt{5}}{\sqrt{1}}$

$= 52 \pm 4.38 = (47.62, 56.38)$

Without σ^2 known, we would have to use the t distribution, and must have at least $n=2$ or else $s^2 = 0 = s$ in the denominator.

7 (a) $H_0: \lambda \leq 8$ (b) $H_1: \lambda > 8$

(c)

$$Z_1 = \frac{\sqrt{n}\,(\bar{X}_n - 8)}{\sqrt{\bar{X}_n}}, \quad Z_2 = \frac{\sqrt{n}\,(\bar{X}_n - 8)}{\sqrt{8}}$$

(d) (Asymptotically) standard normal

(e) $\Omega_0 = \{\lambda : \lambda \leq 8\}$ (f) $\Omega_1 = \{\lambda : \lambda > 8\}$

(g) $z_{1-\alpha}$ (h) $z_{0.95} = 1.645$ (i) $z_{0.99} = 2.326$

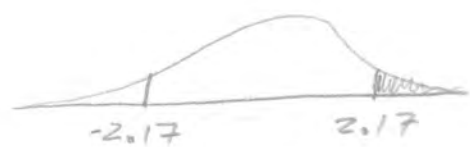
(j) Reject Ho when $Z \geq z_{1-\alpha}$

(k)

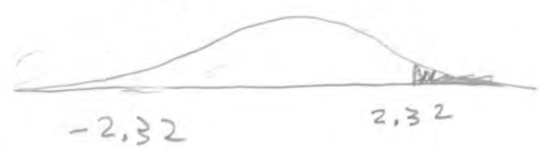
$$Z_1 = \frac{\sqrt{30}\,(9.2-8)}{\sqrt{9.2}} = 2.17$$

$$Z_2 = \frac{\sqrt{30}\,(9.2-8)}{\sqrt{8}} = 2.32$$

(l) $P_1 = \Phi(-2.17) = 0.015$



$P_2 = \Phi(-2.32) = 0.0102$



(m) Yes and Yes

(n) $\lambda > 8$

(o) Yes.

8 6.3.11

(a) $X_1 \cdots X_n \overset{iid}{\sim} \text{Bernoulli}(\theta)$, $\theta = P(\text{2 pips})$

(b) $H_0: \theta = \frac{1}{6}$ (c) $H_1: \theta \neq \frac{1}{6}$

(d) $$Z_1 = \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\bar{X}_n(1-\bar{X}_n)}}, \quad Z_2 = \frac{\sqrt{n}(\bar{X}_n - \theta_0)}{\sqrt{\theta_0(1-\theta_0)}}$$

(e) Both asymptotically standard normal

(f) $\Omega_0 = \{\theta_0\}$ (g) $\Omega_1 = \{\theta \in (0,1) : \theta \neq \theta_0\}$

(h) $\pm z_{1-\frac{\alpha}{2}}$ (i) $\pm z_{0.975} = \pm 1.96$ (j) $\pm z_{0.995} = \pm 2.576$

(k) Reject H_0 if $|Z| > z_{1-\frac{\alpha}{2}}$

(l) $$Z_1 = \frac{\sqrt{30}\left(\frac{1}{3} - \frac{1}{6}\right)}{\sqrt{1/3 * 2/3}} = 1.94$$

$$Z_2 = \frac{\sqrt{30}\left(\frac{1}{3} - \frac{1}{6}\right)}{\sqrt{1/6 * 5/6}} = 2.45$$

(m) $P_1 = 2 * \Phi(-1.94) = 2 * 0.0262 = 0.0524$

$P_2 = 2 * \Phi(-2.45) = 2 * 0.0071 = 0.0142$

(8n) With Z_1, do not reject H_0 : No
With Z_2, reject H_0 : Yes

(o) With Z_1, conclude $\theta = \frac{1}{6}$
With Z_2, conclude $\theta > \frac{1}{6}$

(p) With Z_1, no. With Z_1, Yes. The evidence is inconclusive.

9 (a) $\left.\begin{array}{l} X_1, \ldots, X_{n_1} \overset{iid}{\sim} N(\mu_1, \sigma^2) \\ Y_1, \ldots, Y_{n_2} \overset{iid}{\sim} N(\mu_2, \sigma^2) \end{array}\right\}$ X_i independent of Y_j, $i=1,\ldots,n_1$, $j=1,\ldots,n_2$

(b) The attractiveness ratings are averages (or sums). The CLT suggests they may be approximately normal if the number of judges is substantial.

(c) $\Omega = \{(\mu_1, \mu_2, \sigma^2) \in \mathbb{R}^3 : \sigma^2 > 0\}$

(d) $H_0 : \mu_1 = \mu_2$

(e) $H_1 : \mu_1 \neq \mu_2$

(f) $T = \dfrac{\bar{x} - \bar{y}}{S_p \sqrt{1/n_1 + 1/n_2}}$, $S_p = \sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}}$

$\sim t(n_1+n_2-2)$ under H_0

(9g) $\Omega_0 = \{(\mu_1, \mu_2, \sigma^2) \in \mathbb{R}^3 : \mu_1 = \mu_2,\ \sigma^2 > 0\}$

(h) $\Omega_1 = \{(\mu_1, \mu_2, \sigma^2) \in \mathbb{R}^3 : \mu_1 \neq \mu_2,\ \sigma^2 > 0\}$

(i) $\pm t_{1-\frac{\alpha}{2}}(n_1 + n_2 - 2)$

(j) $\pm t_{0.975}(14) = \pm 2.145$

(k) $\pm t_{0.995}(14) = \pm 2.977$

(l) Reject H_0 when $|T| > t_{1-\frac{\alpha}{2}}(n_1 + n_2 - 2)$

(m) $$s_p^2 = \frac{(9-1)(48.2) + (7-1)(32.7)}{9+7-2} = \frac{581.8}{14} = 41.56$$

$$t = \frac{14.1 - 13.3}{\sqrt{41.56}\sqrt{1/9 + 1/7}} = \frac{0.8}{3.25} = 0.246$$

(n) No.

(o) $\mu_1 = \mu_2$

(p) Nobody.

10 (a) $F = \dfrac{S_1^2}{S_2^2}$

(b) Under H_0: $\dfrac{S_1^2}{S_2^2} \sim F(n_1-1, n_2-1)$

(c) It has previously been shown that for a normal random sample, $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$.
If $W_1 \sim \chi^2(r_1)$ & $W_2 \sim \chi^2(r_2)$ are independent, then $F = \dfrac{W_1/r_1}{W_2/r_2} \sim F(r_1, r_2)$. This is the definition of the F distribution. So

$$W_1 = \frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2(n_1-1) \text{ \& } W_2 = \frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2(n_2-1)$$

They are independent because S_1^2 is a function of $X_1, \ldots, X_{n_1}$ & S_2^2 is a function of $Y_1, \ldots, Y_{n_2}$, and $X_1 \ldots X_{n_1}$ are independent of $Y_1 \ldots Y_{n_2}$. Then if $\sigma_1^2 = \sigma_2^2 = \sigma^2$,

$$F = \frac{\dfrac{\cancel{(n_1-1)}S_1^2}{\cancel{\sigma^2}} \Big/ \cancel{(n_1-1)}}{\dfrac{\cancel{(n_2-1)}S_2^2}{\cancel{\sigma^2}} \Big/ \cancel{(n_2-1)}} = \frac{S_1^2}{S_2^2} \sim F(n_1-1, n_2-1)$$

(10d) $F = \frac{48.2}{32.7} = 1.47$

(e) $F_{0.975}(8,6) = 5.6$ on p. 716

To get $F_{0.025}(8,6)$ note that if

$F = \frac{W_1/r_1}{W_2/r_2} \sim F(r_1, r_2)$, then $\frac{1}{F} = \frac{W_2/r_2}{W_1/r_1} \sim F(r_2, r_1)$

And $P(F < x) = P\left(\frac{1}{F} > \frac{1}{x}\right)$.

So the 0.025 quantile of an $F(8,6)$ is one over the 0.975 quantile of an $F(6,8)$.

From page 718, $F_{0.975}(6,8) = 4.65$, ∴

$F_{0.25}(8,6) = \frac{1}{4.65} = 0.215$ That was a pain.

(f) Reject H_0 if $F < 0.215$ or $F > 5.6$

(g) Since $F = 1.47$, No

(h) Conclude $\sigma_1^2 = \sigma_2^2$

11 (a)(i) $F_{Y_1}(y) = P(Y_1 \le y) = P(\mathrm{Max}(X_i) \le y)$

$= P(\text{All } X_i \le y) = P\left(\bigcap_{i=1}^{n} \{X_i \le y\}\right)$

$\overset{\text{ind}}{=} \prod_{i=1}^{n} P(X_i \le y) = \prod_{i=1}^{n} F(y) = F(y)^n$

(ii) $f_{Y_1}(y) = \frac{d}{dy} F_{Y_1}(y) = n\, F(y)^{n-1} f(y)$

(b) (i) $F_{Y_2}(y) = P(Y_2 \le y) = P(\mathrm{Min}(X_i) \le y)$

$= 1 - P(\mathrm{Min}(X_i) > y) = 1 - P(\text{All } X_i > y)$

$= 1 - P\left(\bigcap_{i=1}^{n} \{X_i > y\}\right) \overset{\text{ind}}{=} 1 - \prod_{i=1}^{n} P(X_i > y)$

$= 1 - (1 - F(y))^n$

(ii) $f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \frac{d}{dy}\left[1 - (1 - F(y))^n\right]$

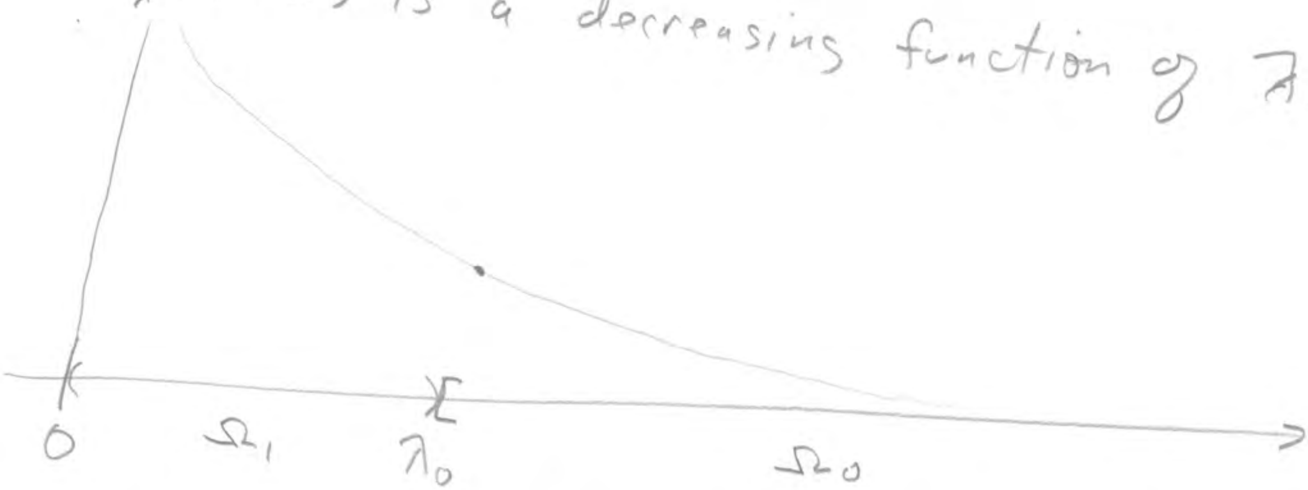
$= -n(1 - F(y))^{n-1}(-f(y))$

$= n(1 - F(y))^{n-1} f(y)$

12 (a) $F_Y(y) = (F(y))^n = \left((1-e^{-\lambda y})I(y>0)\right)^n$
$= (1-e^{-\lambda y})^n I(y>0)$

(b) $P_\lambda(Y \geq k) = 1-(1-e^{-\lambda k})^n$

(c) $\frac{d}{d\lambda} P_\lambda(Y \geq k) = (-1)n(1-e^{-\lambda k})^{n-1}(-1)e^{-\lambda k}(-k)$
$= -nke^{-\lambda k}(1-e^{-\lambda k})^{n-1} < 0$ so
$P_\lambda(Y \geq k)$ is a decreasing function of λ.



So $\max_{\lambda \in \Omega_0} P_\lambda(Y \geq k)$ occurs at $\lambda = \lambda_0$

(12d) Set $P_{\lambda_0}(Y \geq k) = 1-(1-e^{-\lambda_0 k})^n = \alpha$

$\Longrightarrow (1-e^{-\lambda_0 k})^n = 1-\alpha \Longrightarrow 1-e^{-\lambda_0 k} = (1-\alpha)^{1/n}$

$\Longrightarrow e^{-\lambda_0 k} = 1-(1-\alpha)^{1/n}$

$\Longrightarrow -\lambda_0 k = \ln\left(1-(1-\alpha)^{1/n}\right)$

$\Longrightarrow k = -\frac{1}{\lambda_0}\ln\left(1-(1-\alpha)^{1/n}\right)$

(e) $k = -\frac{1}{1/2}\ln\left(1-(1-.05)^{\frac{1}{29}}\right) = -2\ln\left(1-0.95^{\frac{1}{29}}\right)$

$= 12.68$

(f) (i) $P_{\lambda_0}(Y \geq 10.25) = 1-(1-e^{-\frac{1}{2}(10.25)})^{29}$

$= 0.1588 = p\text{-value}$

(ii) If $Y \geq k$ or if $p < \alpha$

(iii) No. $Y < k$ & $p > 0.05$

(iv) $E(X_i) \leq 2 \iff \lambda \geq \frac{1}{2}$

(g) (i) $P_\lambda(Y > k) = P_\lambda(Y > 12.68) = 1-\left(1-e^{-(\lambda)12.68}\right)^{29}$

$= 1-(1-e^{-(0.4)(12.68)})^{29} = 0.1667$

or 0.1588 without rounding error

(ii) $k = 17.28$

Power $= 1-(1-e^{-0.4(17.28)})^{290} = 0.251$ Not too great

(13) We already know $2n\lambda\bar{X}_n \sim \chi^2(2n)$

& $F = \dfrac{W_1/r_1}{W_2/r_2}$ $\quad W_1$ & W_2 independent χ^2s

So if

$$\left.\begin{aligned} X_1,\dots,X_{n_1} &\overset{iid}{\sim} Exp(\lambda_1) \\ Y_1,\dots,Y_{n_2} &\overset{iid}{\sim} Exp(\lambda_2) \end{aligned}\right\} \text{independent}$$

then $W_1 = 2n_1\lambda_1\bar{X}_n \sim \chi^2(2n_1)$ & $W_2 = 2n_2\lambda_2\bar{Y}_n \sim \chi^2(2n_2)$ are independent, and if $H_0: \lambda_1 = \lambda_2$ is true,

$$F = \frac{\cancel{2n_1\lambda_1}\bar{X}_n/\cancel{2n_1}}{\cancel{2n_2\lambda_2}\bar{Y}_n/\cancel{2n_2}} = \frac{\bar{X}_n}{\bar{Y}_n} \sim F(2n_1, 2n_2)$$

Critical values are $F_{\alpha/2}(2n_1, 2n_2)$ & $F_{1-\alpha/2}(2n_1, 2n_2)$