Assignment 5

① (a) iii (b) iii (c) iii (d) v

(e) ii (f) ii

② (a) F (b) T (c) F (d) F

(e) F (f) T (g) F (h) F

(i) T (j) T (k) T (l) F

③ (a) $\ln \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \ln\left(\theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}\right)$

$= \sum_{i=1}^{n} x_i \ln\theta + \left(n-\sum_{i=1}^{n} x_i\right)\ln(1-\theta)$ Bernoulli likelihood

(b) $\ln \prod_{i=1}^{n} \binom{m}{x_i}\theta^{x_i}(1-\theta)^{m-x_i}$

$= \ln\left(\prod_{i=1}^{n}\binom{m}{x_i}\theta^{\sum x_i}(1-\theta)^{nm-\sum x_i}\right)$

$= \sum_{i=1}^{n}\ln\binom{m}{x_i} + \sum_{i=1}^{n} x_i \ln\theta + \left(nm-\sum_{i=1}^{n} x_i\right)\ln(1-\theta)$ Binomial likelihood

(c) $\ln \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \ln\left[\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right]$

$= -n\lambda + \sum_{i=1}^{n} x_i \ln\lambda - \sum_{i=1}^{n}\ln x_i!$ Poisson likelihood

(d) $\ln \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \ln\left[\theta^{n}(1-\theta)^{\sum x_i - n}\right]$

$= n\ln\theta + \left(\sum_{i=1}^{n} x_i - n\right)\ln(1-\theta)$

one version of the geometric likelihood

(3e) $\ln \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \ln\left[\theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}\right]$

$= -n \ln\theta - \frac{1}{\theta}\sum_{i=1}^{n} x_i$ One version of the exponential likelihood

(f) $\ln \prod_{i=1}^{n} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} e^{-x_i/\beta} x_i^{\alpha-1}$

$= \ln\left[\beta^{-\alpha}\Gamma(\alpha)^{-n} e^{-\frac{1}{\beta}\sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}\right]$

$= -\alpha \ln\beta - n\ln\Gamma(\alpha) - \frac{1}{\beta}\sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\ln x_i$

One version of the Gamma likelihood

(g) $\ln \prod_{i=1}^{n} \frac{1}{2^{\gamma/2}\Gamma(\gamma/2)} e^{-x_i/2} x_i^{\gamma/2-1}$

$= \ln\left[2^{-n\gamma/2}\Gamma\left(\frac{\gamma}{2}\right)^{-n} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i}\left(\prod_{i=1}^{n} x_i\right)^{\frac{\gamma}{2}-1}\right]$

$= -\frac{n\gamma}{2}\ln 2 - n\ln\Gamma\left(\frac{\gamma}{2}\right) - \frac{1}{2}\sum_{i=1}^{n} x_i + \left(\frac{\gamma}{2}-1\right)\sum_{i=1}^{n}\ln x_i$

Chi-squared likelihood

(h) $\ln \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$

$= \ln\left[\sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}\right]$

$= -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$ Normal likelihood

(i) $\prod_{i=1}^{n} \frac{1}{\beta-\alpha} I(\alpha \le x_i \le \beta) = (\beta-\alpha)^{-n}\prod_{i=1}^{n} I(\alpha \le x_i \le \beta)$

$= (\beta-\alpha)^{n} I(\alpha \le \mathrm{Min}(x_i))\, I(\mathrm{Max}(x_i) \le \beta)$

(4) (a) From Question 3c, $\ell(\lambda) = -n\lambda + (\sum_{i=1}^{n} x_i)\ln\lambda - \sum_{i=1}^{n}\ln x_i!$

$$\ell'(\lambda) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} - 0 \overset{set}{=} 0 \Rightarrow \frac{\sum x_i}{\lambda} = n$$

$$\Rightarrow \lambda = \bar{x}_n$$

(b) $$\ell''(\lambda) = \frac{d}{d\lambda}\left(-n + (\sum x_i)\lambda^{-1}\right) = \left(\sum_{i=1}^{n} x_i\right)(-\lambda^{-2})$$

$$= \frac{-\sum_{i=1}^{n} x_i}{\lambda^2} < 0$$ Negative CCD $\cap$ Max

So $\hat{\lambda} = \bar{x}_n$

(c) $\hat{\lambda} = 4.2$

(d) By LLN, $\bar{X}_n \xrightarrow{P} \lambda = \sigma^2$, so by CLT,

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} = \frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\bar{X}_n}}$$

$\xrightarrow{d} Z \sim N(0,1)$, ∴ a $(1-\alpha)100\%$ CI is

$$\bar{x}_n \pm z_{1-\frac{\alpha}{2}} \frac{\hat{\sigma}_n}{\sqrt{n}} = \bar{x}_n \pm z_{1-\alpha/2} \frac{\sqrt{\bar{x}_n}}{\sqrt{n}}$$

$$= 4.2 \pm 1.96 \frac{\sqrt{4.2}}{7} = 4.2 \pm 0.574$$

$$= (3.626, 4.774)$$

(5) Exercise 6.2.2 : See lecture of Jan. 30

Exercise 6.2.3 : By the invariance principle $\widehat{\theta^2} = \bar{X}_n^2$

(6) (a) Exercise 6.2.5

$$\ell(\theta) = \ln \prod_{i=1}^{n} \frac{\theta^{\alpha_0}}{\Gamma(\alpha_0)} e^{-\theta x_i} x_i^{\alpha_0 - 1}$$

$$= \ln\left[\theta^{n\alpha_0}\Gamma(\alpha_0)^{-n} e^{-\theta\sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{\alpha_0 - 1}\right]$$

$$= n\alpha_0 \ln\theta - n\ln\Gamma(\alpha_0) - \theta\sum_{i=1}^{n} x_i + \ln\left(\prod_{i=1}^{n} x_i\right)^{\alpha_0 - 1}$$

$$\ell'(\theta) = \frac{n\alpha_0}{\theta} - \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0 \implies \frac{n\alpha_0}{\theta} = \sum_{i=1}^{n} x_i$$

$$\implies \theta = \frac{n\alpha_0}{\sum_{i=1}^{n} x_i} = \frac{\alpha_0}{\bar{x}_n}$$

$$\ell''(\theta) = \frac{d}{d\theta}\left(n\alpha_0\theta^{-1} - \sum_{i=1}^{n} x_i\right) = \frac{-n\alpha_0}{\theta^2}$$ neg ccd

MAX

So $\hat{\theta} = \frac{\alpha_0}{\bar{x}_n}$

(b) For $\alpha_0 = 5$ and $\bar{x}_n = 3.455$

$$\hat{\theta} = \frac{5}{3.455} = 1.45$$

(7) (a) $\ell(\theta) = \ln \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1} = \ln\left[\theta^n (1-\theta)^{\sum_{i=1}^{n} x_i - n}\right]$

$= n \ln\theta + \left(\sum_{i=1}^{n} x_i - n\right)\ln(1-\theta)$

$$\ell'(\theta) = \frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i - n}{1-\theta}(-1) = \frac{n}{\theta} - \frac{\sum x_i - n}{1-\theta} \stackrel{set}{=} 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i - n}{1-\theta} \Rightarrow n - \cancel{n\theta} = \theta\sum_{i=1}^{n} x_i - \cancel{n\theta}$$

$$\Rightarrow \theta = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}_n}$$

$$\ell''(\theta) = \frac{d}{d\theta}\left(n\theta^{-1} - \left(\sum x_i - n\right)(1-\theta)^{-1}\right)$$

$$= -n\theta^{-2} - \left(\sum x_i - n\right)(-1)(1-\theta)^{-2}\cdot(-1)$$

$$= -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i - n}{(1-\theta)^2} < 0$$

Because $x_i \geq 1$ $\Rightarrow \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{n} 1 = n$

Concave down, Max, and

$$\hat{\theta} = \frac{1}{\bar{x}_n}$$

(b) $\bar{x}_n = 4.85$, $\hat{\theta} = \frac{1}{4.85} \doteq 0.206$

$\boxed{6}$

(8) (a) $\ell(\theta) = \ln \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta} = \ln\left[\theta^{-n} e^{-\frac{1}{\theta}\Sigma x_i}\right]$

$= -n \ln \theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i = -n \ln\theta - (\sum_{i=1}^{n} x_i)\theta^{-1}$

$\ell'(\theta) = \frac{-n}{\theta} - (\sum_{i=1}^{n} x_i)(-\theta^{-2}) = \frac{-n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \overset{\text{set}}{=} 0$

$\Rightarrow \frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i}{\theta^2} \Rightarrow n = \frac{\Sigma x_i}{\theta} \Rightarrow \theta = \bar{x}_n$

$\ell''(\theta) = \frac{d}{d\theta}\left(-n\theta^{-1} + (\Sigma x_i)\theta^{-2}\right)$

$= -n(-1)\theta^{-2} + n\bar{x}(-2)\theta^{-3}$

$= \frac{n}{\theta^2} - \frac{2n\bar{x}}{\theta^3}$ Evaluate at $\theta = \bar{x}$

$= \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} = \frac{n - 2n}{\bar{x}^2} < 0$ neg ccd $\cap$ MAX

and $\hat{\theta} = \bar{x}_n$

(b) $\bar{x} = \hat{\theta} = 1.518$

(9) (a) $E(X) = \int_\theta^\infty x\, 2 e^{-2(x-\theta)}\, dx$ Don't integrate by parts!

$u = x - \theta$
$du = dx$
$x = u + \theta$

x	$u = x - \theta$
∞	∞
θ	0

$$E(X) = \int_0^\infty (u+\theta) \underbrace{2 e^{-2u}}_{\text{Exponential}(\lambda=2)} du$$

$$= \int_0^\infty u\, 2e^{-2u}\, du + \theta \underbrace{\int_0^\infty 2e^{-2u}\, du}_{=1}$$

$= \frac{1}{2} + \theta$ so set $\bar{x} = \frac{1}{2} + \hat{\theta}$ and

$$\hat{\theta}_{MOM} = \bar{x} - \frac{1}{2}$$

(b) $$L(\theta) = \prod_{i=1}^{n} 2 e^{-2(x_i - \theta)}\, I(x_i \geq \theta)$$

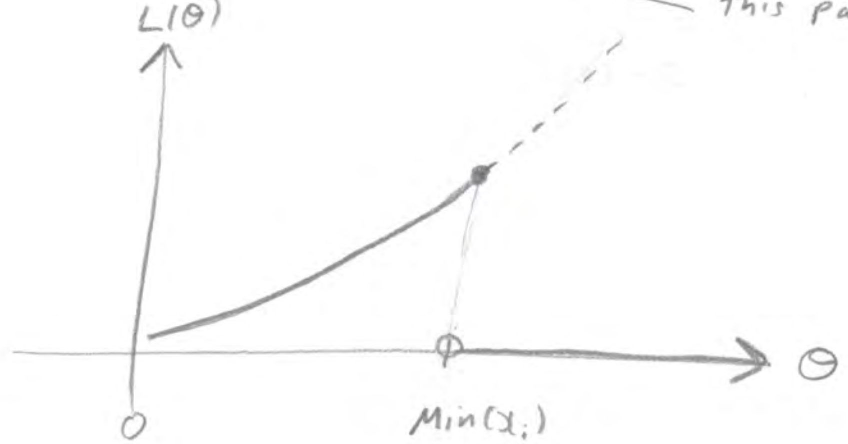
$$= 2^n e^{-2(\Sigma x_i - n\theta)} \prod_{i=1}^{n} I(x_i \geq \theta)$$

$$= 2^n e^{-2n\bar{x}} e^{2n\theta}\, I(\mathrm{Min}(x_i) \geq \theta)$$

(9b) continued

$$L(\theta) = 2^n e^{-2n\bar{x}} \underbrace{e^{2n\theta}} I(\theta \le Min(x_i))$$

This part is increasing in θ



And MLE is $\hat{\theta} = Min(x_i)$

(c) To check unbiased, need the distribution of $\hat{\theta} = Min(X_i)$. For now, call it $Y = Min(X_i)$

$$f_{\hat{\Theta}}(\hat{\theta}) = f_Y(\overset{y \ge \theta}{y}) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y)$$

$$= \frac{d}{dy} P(Min(X_i) \le y) = \frac{d}{dy}\left(1 - P(Min\, X_i > y)\right)$$

$$= -\frac{d}{dy} P(All\ X_i > y) = -\frac{d}{dy} P\left(\bigcap_{i=1}^{n} \{X_i > y\}\right)$$

$$\overset{ind}{=} -\frac{d}{dy} \prod_{i=1}^{n} P(X_i > y) = -\frac{d}{dy} \prod_{i=1}^{n} (1 - F_{X_i}(y))$$

$$\overset{ident\ dist}{=} -\frac{d}{dy} (1 - F_X(y))^n = -n(1 - F_X(y))^{n-1} f_X(y)(-1)$$

$$= n(1 - F_X(y))^{n-1} f_X(y)$$

(9c) Continued Have $f_Y(y|\theta) = n\left(1-F_X(y|\theta)\right)^{n-1} f_X(y|\theta)$

Need $F_X(y|\theta)$, or better, for $y \geq \theta$,

$$1 - F_X(y|\theta) = \int_y^\infty 2e^{-2(x-\theta)}\,dx$$

$$= \int_y^\infty 2e^{-2x} e^{2\theta}\,dx = e^{2\theta} \underbrace{\int_y^\infty 2e^{-2x}\,dx}_{\text{Exponential}(\lambda=2)}$$

$$= e^{2\theta} e^{-2y} = e^{-2(y-\theta)}$$

So $f_Y(y|\theta) = n e^{-2(y-\theta)(n-1)}\, 2e^{-2(y-\theta)}\, I(y \geq \theta)$

$$= 2n e^{-2n(y-\theta)}\, I(y \geq \theta)$$

And so

$$E(Y) = E(\hat{\Theta}_n) = \int_\theta^\infty y\, 2n e^{-2n(y-\theta)}\,dy$$

$u = y - \theta \iff y = u + \theta$, $dy = du$

y	$u = y - \theta$
∞	∞
θ	0

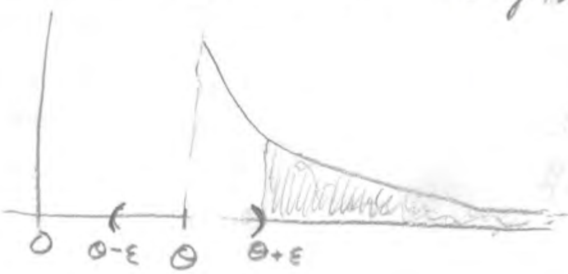
$$= \int_0^\infty (u+\theta)\, 2n e^{-2nu}\,du$$

$= E(U + \theta)$ where $U \sim \text{Exp}(2n)$

$= \frac{1}{2n} + \theta \neq \theta$ Biased

(9d) Can use either the definition or the variance rule

<u>Definition</u> Let $\varepsilon > 0$ be given. $\lim_{n\to\infty} P\{|\hat{\Theta}_n - \theta| \geq \varepsilon\}$



$$= \lim_{n\to\infty}\left(1 - P(\theta - \varepsilon < \hat{\Theta}_n < \theta + \varepsilon)\right)$$

$$= \lim_{n\to\infty} P(\hat{\Theta}_n \geq \theta + \varepsilon) = \lim_{n\to\infty}\left(1 - F_{\hat{\Theta}_n}(\theta + \varepsilon \mid \theta)\right)$$

$$\overset{(9c)}{=} \lim_{n\to\infty} e^{-2n(\theta+\varepsilon-\theta)} = \lim_{n\to\infty} e^{-2n\varepsilon}$$

$$= \lim_{n\to\infty} \frac{1}{e^{2n\varepsilon}} = 0 \quad \text{So } \hat{\Theta}_n \xrightarrow{P} \theta \text{ : } \underline{\text{Consistent}}$$

<u>Variance Rule</u>

i) $\lim_{n\to\infty} E(\hat{\Theta}_n) = \lim_{n\to\infty}\left(\frac{1}{2n} + \theta\right) = 0 + \theta = \theta$

ii) To check $\lim_{n\to\infty} Var(\hat{\Theta}_n) = 0$, use

$$Var(\hat{\Theta}_n) = E(\hat{\Theta}_n^2) - \left(E(\hat{\Theta}_n)\right)^2$$

$$E(\hat{\Theta}_n^2) = \int_\theta^\infty y^2\, 2n\, e^{-2n(y-\theta)}\, dy$$

Same change of variables: $u = y - \theta$, $y = u + \theta$, $dy = du$

$$= \int_0^\infty (u+\theta)^2\, 2n\, e^{-2nu}\, du$$

y	$u = y - \theta$
∞	∞
θ	0

$$= E(U+\theta)^2, \quad U \sim Exp(2n)$$

(9d) continued

$$E(U+\theta)^2 = E(U^2+2U\theta+\theta^2)$$
$$= E(U^2)+2\theta E(U)+\theta^2$$
$$= E(U^2)+2\theta\left(\frac{1}{2n}\right)+\theta^2$$

(using $Var(U) = E(U^2) - (E(U))^2 \Rightarrow E(U^2) = Var(U) + (E(U))^2$,)

$$= \frac{1}{(2n)^2} + \left(\frac{1}{2n}\right)^2 + \frac{\theta}{n} + \theta^2$$

$$= E(\hat{\Theta}_n^2) \text{, so}$$

$$Var(\hat{\Theta}_n) = E(\hat{\Theta}_n^2) - \left[E(\hat{\Theta}_n)\right]^2$$
$$= \frac{2}{4n^2} + \frac{\theta}{n} + \theta^2 - \left[\theta + \frac{1}{2n}\right]^2$$
$$= \frac{1}{2n^2} + \frac{\theta}{n} + \theta^2 - \left(\theta^2 + 2\cdot\frac{\theta}{2n} + \frac{1}{4n^2}\right)$$
$$= \frac{1}{2n^2} + \cancel{\frac{\theta}{n}} + \cancel{\theta^2} - \cancel{\theta^2} - \cancel{\frac{\theta}{n}} - \frac{1}{4n^2}$$
$$= \frac{1}{4n^2}$$

And $\lim_{n\to\infty} Var(\hat{\Theta}_n) = \lim_{n\to\infty} \frac{1}{4n^2} = 0$, so that $\hat{\Theta}_n \xrightarrow{P} \theta$ by the variance rule and $\hat{\Theta}_n$ is consistent for θ.

Using the definition is easier. Even easier than that is to realize that the distribution of $\hat{\Theta}_n$ is $Exp(2n)$ shifted to the right by θ, so $Var(\hat{\Theta}_n) = \frac{1}{(2n)^2} \to 0$

(9e) $\hat{\theta}_{MOM} = \bar{x} - \frac{1}{2} = 6.89 - 0.5 = 6.49$

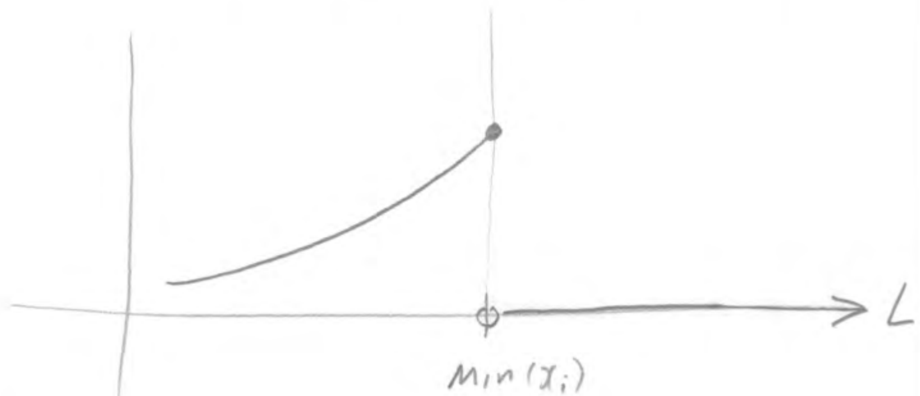
$\hat{\theta}_{MLE} = Min(x_i) = 6.11$

(10) See lecture notes from Thurs. Jan 30 & Thurs. Feb 6.

(11) $L(L, R) = \prod_{i=1}^{n} \frac{1}{R-L} I(L \le x_i \le R) = \left(\frac{1}{R-L}\right)^n \prod_{i=1}^{n} I(L \le x_i \le R)$

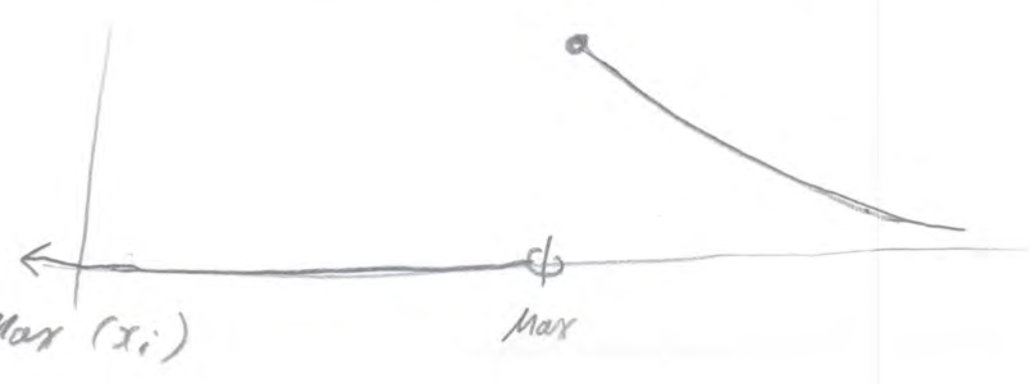
$= \left(\frac{1}{R-L}\right)^n I(Min(x_i) \ge L)\, I(Max(x_i) \le R)$

$\frac{\partial}{\partial L}(R-L)^{-n} = -n(R-L)^{-n-1} \cdot (-1) > 0$, so for every $R > L$, likelihood is increasing in $L \le Min(x_i)$



And $\hat{L} = Min(x_i)$

$\frac{\partial}{\partial R}(R-L)^{-n} = -n(R-L)^{-n-1} < 0$ so for every $L < R$ likelihood is decreasing in $R \ge Max(x_i)$



And $\hat{R} = Max(x_i)$

(12) 6.2.11 $\widehat{\mu^3} = \hat{\mu}^3 = 3.2^3 = 32.768$

by the invariance principle

(13) 6.2.12 $l(\sigma) = \ln \prod_{i=1}^{n} \sigma^{-1}(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu_0)^2}$

$= \ln\left[\sigma^{-n}(2\pi)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}\right]$

$= -n\ln\sigma - \frac{n}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2(\sigma^{-2})$

$l'(\sigma) = \frac{-n}{\sigma} - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu_0)^2(-2)\sigma^{-3}$

$= \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu_0)^2 \overset{\text{set}}{=} 0$

$\Longrightarrow n = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2 \Longrightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$

$l''(\sigma) = \frac{d}{d\sigma}\left(-n\sigma^{-1} + \sigma^{-3}\sum_{i=1}^{n}(x_i-\mu_0)^2\right)$

$= \frac{n}{\sigma^2} - 3\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sigma^4}$ Setting $\hat{\sigma}^2 = \frac{\sum(x_i-x_0)^2}{n}$

Evaluate at $\sigma^2 = \hat{\sigma}^2$, get

$\frac{n}{\hat{\sigma}^2} - 3\frac{n\hat{\sigma}^2}{\hat{\sigma}^4} = -2\frac{n}{\hat{\sigma}^2}$ neg $\underset{\text{MAX}}{\text{CCD}}$ $\cap$

So the MLE is $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{n}$

Different from $\hat{\sigma}^2 = \frac{\sum(x_i-\bar{x})^2}{n}$ from the full MLE.
I don't know why they call it the "plug-in" estimate.

(Q13 continued) 6.2.14
I would check $l(\theta)$ at all 3 points, and let $\hat{\theta}$ be the value with $l(\theta)$ the largest.

6.2.16 No. It's the probability of getting the observed data when $\theta = 2$.

(14) (a) $E_\theta(X_i = m_i\theta)$, so minimize $Q = \sum_{i=1}^{n}(x_i - m_i\theta)^2$

$$\frac{\partial Q}{\partial \theta} = \sum_{i=1}^{n} 2(x_i - m_i\theta)(-m_i) = -2\left(\sum_{i=1}^{n} m_i x_i - \theta \sum_{i=1}^{n} m_i^2\right)$$

$$\overset{set}{=} 0 \Rightarrow \sum_{i=1}^{n} m_i x_i = \theta \sum_{i=1}^{n} m_i^2 \Rightarrow \theta = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i^2}$$

$$\frac{\partial^2 Q}{\partial \theta^2} = (-2)\left(-\sum_{i=1}^{n} m_i^2\right) = 2\sum_{i=1}^{n} m_i^2 > 0 \quad \text{ccup } \cup \text{ min}$$

so $\hat{\theta}_{LS} = \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i^2}$

(b) $$E(\hat{\theta}_{LS}) = E\left(\frac{\sum_{i=1}^{n} m_i X_i}{\sum_{i=1}^{n} m_i^2}\right) = \frac{\sum_{i=1}^{n} m_i E(X_i)}{\sum_{i=1}^{n} m_i^2}$$

$$= \frac{\sum_{i=1}^{n} m_i\, m_i \theta}{\sum_{i=1}^{n} m_i^2} = \theta \frac{\sum_{i=1}^{n} m_i^2}{\sum_{i=1}^{n} m_i^2} = \theta \quad \underline{\text{unbiased}}$$

(14c) $\ell(\theta) = \ln \prod_{i=1}^{n} \binom{m_i}{x_i} \theta^{x_i} (1-\theta)^{m_i - x_i}$

$$= \ln\left[\prod_{i=1}^{n} \binom{m_i}{x_i} \theta^{\Sigma x_i} (1-\theta)^{\Sigma m_i - \Sigma x_i} \right]$$

$$= \ln \prod_{i=1}^{n} \binom{m_i}{x_i} + \left(\sum_{i=1}^{n} x_i\right) \ln \theta + \left(\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} x_i\right) \ln(1-\theta)$$

$$\ell'(\theta) = \frac{\Sigma x_i}{\theta} - \frac{\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} x_i}{1-\theta} \overset{set}{=} 0$$

$$\Rightarrow \frac{\Sigma x_i}{\theta} = \frac{\Sigma m_i - \Sigma x_i}{1-\theta}$$

$$\Rightarrow \sum_{i=1}^{n} x_i \cancel{- \theta \sum_{i=1}^{n} x_i} = \theta \sum_{i=1}^{n} m_i \cancel{- \theta \sum_{i=1}^{n} x_i}$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} m_i}$$

$$\ell''(\theta) = \frac{d}{d\theta}\left(\theta^{-1} \sum_{i=1}^{n} x_i - (1-\theta)^{-1} \left(\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} x_i \right) \right)$$

$$= -\theta^{-2} \sum_{i=1}^{n} x_i - (1-\theta)^{-2}(-1)\left(\sum_{i=1}^{n} m_i - \sum_{i=1}^{n} x_i \right)$$

$$= \frac{\sum_{i=1}^{n} x_i}{\theta^2} + \frac{\Sigma m_i - \Sigma x_i}{(1-\theta)^2} > 0$$

Bec $m_i \geq x_i$

$\Rightarrow \Sigma m_i \geq \Sigma x_i$

But if they are $=$, $\Sigma x_i > 0$

CCUP MAX

So $$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} m_i}$$

(d) $E(\hat{\Theta}_n) = E\left(\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n m_i}\right) = \frac{\sum_{i=1}^n E(X_i)}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_i \theta}{\sum_{i=1}^n m_i}$

$= \theta \frac{\cancel{\sum_{i=1}^n m_i}}{\cancel{\sum_{i=1}^n m_i}} = \theta$ unbiased

(e) First $m_i \geq 1 \Rightarrow \sum_{i=1}^n m_i \geq \sum_{i=1}^n 1 = n \Rightarrow \frac{1}{\sum_{i=1}^n m_i} \leq \frac{1}{n}$

$\Longrightarrow \frac{\theta(1-\theta)}{\sum_{i=1}^n m_i} \leq \frac{\theta(1-\theta)}{n}$

<u>Now</u> using the variance rule,

(i) $\lim_{n\to\infty} E(\hat{\Theta}_n) = \lim_{n\to\infty} \theta = \theta$

(ii) $Var(\hat{\Theta}_n) = Var\left(\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n m_i}\right)$

$= \frac{1}{\left(\sum_{i=1}^n m_i\right)^2} Var\left(\sum_{i=1}^n X_i\right) \overset{\text{ind}}{=} \frac{1}{\left(\sum_{i=1}^n m_i\right)^2} \sum_{i=1}^n Var(X_i)$

$= \frac{1}{\left(\sum_{i=1}^n m_i\right)^2} \sum_{i=1}^n m_i \theta(1-\theta) = \frac{\theta(1-\theta)}{\sum_{i=1}^n m_i}$, so

$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} Var(\hat{\Theta}_n) = \lim_{n\to\infty} \frac{\theta(1-\theta)}{\sum_{i=1}^n m_i} \leq \lim_{n\to\infty} \frac{\theta(1-\theta)}{n} = 0$

So $\lim_{n\to\infty} Var(\hat{\Theta}_n) = 0$ by squeeze, & by the variance rule $\hat{\Theta}_n \xrightarrow{P} \theta$ <u>consistent</u>

(14f)

i	m_i	x_i	m_i^2	$m_i x_i$
1	10	4	100	40
2	13	7	169	91
3	5	0	25	0
4	18	14	324	252
5	17	6	289	102
6	6	4	36	24
7	14	5	196	70
8	5	4	25	20
9	10	3	100	30
10	14	4	196	56
SUM	112	51	1460	685

$$\hat{\Theta}_{LS} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i^2} = \frac{685}{1460} = 0.469$$

$$\hat{\Theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} m_i} = \frac{51}{112} = 0.455$$

15 (a) $E(Y_i) = E(\beta x_i + E_i) = \beta x_i + E(E_i) = \beta x_i + 0$

(b) $Var(Y_i) = Var(\beta x_i + E_i) = Var(E_i) = \sigma^2$

(c) See lecture notes of Tues. Jan 28.

(d) " " " " " " "

(e) " " " " " " "

(f)

	x_i	x_i^2	y_i	$x_i y_i$
	1	1	6	6
	6	36	37	222
	3	9	12	36
	7	49	52	364
	2	4	4	8
sum	19	99	111	636

$$\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{636}{99} = 6.42$$

(15g) i. $Y_i \sim N(\beta x_i, \sigma^2)$

ii) $\ell(\beta, \sigma^2) = \log \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i - x_i\beta)^2}$

$= \log\left[\sigma^{-n}(2\pi)^{-n/2} \operatorname{Exp} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)^2\right]$

$= -n\log\sigma - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)^2$

$$\frac{\partial \ell}{\partial \beta} = \frac{-1}{\cancel{2}\sigma^2}\sum_{i=1}^{n}\cancel{2}(y_i - \beta x_i)(-x_i)$$

$$= \frac{-1}{\sigma^2}\sum_{i=1}^{n}(x_i y_i - \beta x_i^2) = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} x_i y_i - \beta\sum_{i=1}^{n} x_i^2\right)$$

$$\stackrel{\text{set}}{=} 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = \beta\sum_{i=1}^{n} x_i^2 \Rightarrow \beta = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

FOR every $\sigma^2 > 0$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{1}{\sigma^2}\left(-\sum_{i=1}^{n} x_i^2\right) = \frac{-\sum_{i=1}^{n} x_i^2}{\sigma^2} < 0 \quad \text{CCD } \underset{\text{MAX}}{\cap}$$

So that for every $\sigma^2 > 0$, likelihood is maximized at $\hat{\beta} = \frac{\sum x_i y_i}{\sum_{i=1}^{n} x_i^2}$

Set $\beta = \hat{\beta}$ & maximize over σ, get $\hat{\sigma}^2$ by invariance

(15 g ii) continued

$$\frac{\partial l}{\partial \sigma}\Big|_{\beta=\hat\beta} = \frac{\partial}{\partial \sigma}\left(-n\log\sigma - \frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2\sigma^{-2}\right)$$

$$= \frac{-n}{\sigma} - \frac{1}{\cancel{2}}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2(-\cancel{2})\sigma^{-3}$$

$$= \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2 \overset{\text{set}}{=} 0$$

$$\Longrightarrow n = \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2$$

$$\Longrightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat\beta x_i)^2$$

And

$$\frac{\partial^2 l}{\partial^2 \sigma}\Big|_{\beta=\hat\beta} = \frac{\partial}{\partial\sigma}\left(-n\sigma^{-1} + \sigma^{-3}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2\right)$$

$$= (-n)(-\sigma^{-2}) - 3\sigma^{-4}\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2$$

$$= \frac{n}{\sigma^2} - 3\sum_{i=1}^{n}(y_i - x_i\hat\beta)^2/\sigma^4$$

Evaluate at $\hat\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat\beta x_i)^2$, get

$$= \frac{n}{\hat\sigma^2} - 3\frac{n\hat\sigma^2}{\hat\sigma^4} = \frac{-2n}{\hat\sigma^2} < 0 \quad \text{CCD} \cap$$

MAX

And MLEs are

$$\hat\beta = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}, \quad \hat\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat\beta x_i)^2$$

(15g iii) Since Y_i are normal, $\hat{\beta}$ is normal, with expected value & variance

$$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right) = \frac{\sum_{i=1}^{n} x_i E(Y_i)}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i x_i \beta}{\sum_{i=1}^{n} x_i^2}$$

$$= \beta \frac{\sum x_i^2}{\sum x_i^2} = \beta \text{ , and}$$

$$Var(\hat{\beta}) = Var\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum x_i^2}\right) = \frac{1}{\left(\sum_{i=1}^{n} x_i^2\right)^2} Var \sum_{i=1}^{n} x_i Y_i$$

$$\overset{ind}{=} \frac{1}{(\sum x_i^2)^2} \sum_{i=1}^{n} x_i^2 Var(Y_i) = \frac{\sum_{i=1}^{n} x_i^2 \sigma^2}{\left(\sum_{i=1}^{n} x_i^2\right)^2}$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2} \text{ , so}$$

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

(15g iv) Because $\hat{\beta} \sim N\left(\beta, \frac{\sigma_0^2}{\sum_{i=1}^{n} x_i^2}\right)$,

$$Z = \frac{\hat{\beta} - \beta}{\sqrt{\sigma_0^2 / \sum_{i=1}^{n} x_i^2}} \sim N(0,1) \text{, and}$$

$$1 - \alpha = P\left(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}\right)$$

$$= P\left(-z_{1-\alpha/2} < \frac{\hat{\beta} - \beta}{\sqrt{\sigma_0^2 / \sum x_i^2}} < z_{1-\alpha/2}\right)$$

$$= P\left(-z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}} < \hat{\beta} - \beta < z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}}\right)$$

$$= P\left(-\hat{\beta} - z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}} < -\beta < -\beta + z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}}\right)$$

=

$$= P\left(\hat{\beta} - z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}} < \beta < \hat{\beta} + z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum x_i^2}}\right)$$

or

$$\hat{\beta} \pm z_{1-\alpha/2} \frac{\sigma_0}{\sqrt{\sum_{i=1}^{n} x_i^2}}$$

v. $6.42 \pm 1.96 \frac{2}{\sqrt{99}} = 6.42 \pm 0.394 = (6.03, 6.81)$

16 (a) See lecture of Tues Jan 28

(b) $Y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$, so maximize

$\ell(\beta_0, \beta_1, \sigma^2)$ over β_0 & β_1

$$= \ln \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2}$$

$$= \ln\left[\sigma^{-n}(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}\right]$$

$$= -n\ln\sigma - \frac{n}{2}\ln 2\pi - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2$$

$$= -n\ln\sigma - \frac{n}{2}\ln 2\pi - \frac{1}{2\sigma^2}Q(\beta_0, \beta_1), \text{ where}$$

$Q(\beta_0, \beta_1)$ is the least squares criterion.

Thus maximizing ℓ over (β_0, β_1) is the same as minimizing Q — same solution.