Assignment 2

① (a) $\Omega = \{\theta : 0 < \theta < 1\}$

(b) $E(\hat{\Theta}_n) = E\left(\frac{1}{3}\bar{X}_n\right) = \frac{1}{3}E(\bar{X}_n) = \frac{1}{3}E(X_i)$

$= \frac{1}{3}\,3\theta = \theta$ unbiased.

(c) By LLN, $\bar{X}_n \xrightarrow{P} E(X_i) = 3\theta$. Then by Continuous mapping $\frac{1}{3}\bar{X}_n \xrightarrow{P} \frac{1}{3}3\theta = \theta$ consistent.

② (a) $\Omega = \{\theta : \theta > 0\}$

(b) $E(X_i) = \int_0^1 x\,\theta x^{\theta-1}\,dx = \theta\int_0^1 x^{\theta}\,dx$

$= \theta\,\frac{x^{\theta+1}}{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1}$

So $E(\bar{X}_n) = \frac{\theta}{\theta+1} \neq \theta$, biased

(c) By LLN $\bar{X}_n \xrightarrow{P} \frac{\theta}{\theta+1} \neq \theta$ inconsistent

(3) (a) $E(S^2) = E\left(\frac{\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}{n-1} \right)$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n} (X_i - \mu + \mu - \bar{X}_n)^2 \right)$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} \left((X_i-\mu)^2 + 2(X_i-\mu)(\mu-\bar{X}_n) + (\mu-\bar{X}_n)^2 \right) \right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n}(X_i-\mu)^2 + 2\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n}\mu\right)(\mu-\bar{X}_n) + \sum_{i=1}^{n}(\mu-\bar{X}_n)^2 \right]$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n}(X_i-\mu)^2 + 2(n\bar{X}_n - n\mu)(\mu-\bar{X}_n) + n(\bar{X}_n-\mu)^2 \right)$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n}(X_i-\mu)^2 - 2n(\bar{X}_n-\mu)^2 + n(\bar{X}_n-\mu)^2 \right)$$

$$= \frac{1}{n-1} E\left(\sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}_n-\mu)^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} E(X_i-\mu)^2 - n\,E(\bar{X}_n-\mu)^2 \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} \mathrm{Var}(X_i) - n\,\mathrm{Var}(\bar{X}_n) \right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} \sigma^2 - n\frac{\sigma^2}{n} \right)$$

$$= \frac{1}{n-1} \left(n\sigma^2 - \sigma^2 \right) = \sigma^2 \frac{n-1}{n-1}$$

$$= \sigma^2$$ unbiased

You could skip some steps

3b $E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right)=\frac{1}{n}\sum_{i=1}^{n}E(X_i-\mu)^2$

$= \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \frac{1}{n}n\sigma^2 = \sigma^2$ unbiased

(c) Let $Y_i=(X_i-\mu)^2$. Then $\hat{\sigma}_n^2 = \bar{Y}_n \xrightarrow{P} E(Y_i)=\sigma^2$

(d) The problem does not specify $Var(Y_i)$ exists, a condition that is needed if the Law of Large Numbers is proved using the Variance Rule. However,

(i) This is a highly technical issue, and somewhat artifical. In practice, all random variables are bounded and all moments exist.

(ii) There is a stronger version of the (weak) Law of Numbers that does not require the existence of a variance. The proof is advanced.

(4) (a) $\Omega = \{\lambda : \lambda > 0\}$

(b) $E(\bar{X}_n) = \lambda$ and by LLN, $\bar{X}_n \xrightarrow{p} \lambda$

(c) $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$. As shown in Q3a, S_n^2 is unbiased for $Var(X_i) = \lambda$.

(d) $$S_n^2 = \left(\frac{n}{n-1}\right) \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \left(\frac{n}{n-1}\right) \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)$$

$$= \left(\frac{n}{n-1}\right) \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\bar{X}_n\bar{X}_n + \frac{1}{n}\sum_{i=1}^{n} \bar{X}_n^2\right)$$

$$= \left(\frac{n}{n-1}\right) \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2\right)$$

$$= \left(\frac{n}{n-1}\right) \left(\frac{1}{n} \sum_{i=1}^{n} X_i^2 - \bar{X}_n^2\right),$$ which, by LLN and continuous mapping

$$\xrightarrow{p} 1 \cdot \left(E(X_i^2) - (E(X_i))^2\right) = \sigma^2$$

(e) (i) $E\left(\frac{1}{2}(X_1 + X_2)\right) = \frac{1}{2}\left(E(X_1) + E(X_2)\right)$
$= \frac{1}{2}(\lambda + \lambda) = \lambda$ unbiased

(ii) Not consistent, because $\hat{\lambda}$ has the same non-degenerate distribution on $\{0, 1, \ldots\}$ for all n

(iii) It's silly because it throws away information.

5 (a) $\Omega = \{\theta : \theta > 0\}$

(b) For $0 < x < \theta$, $F_{X_i}(x|\theta) = \int_0^x \frac{1}{\theta}\, dt = \frac{x}{\theta}$,

So $$F_{X_i}(x|\theta) = \frac{x}{\theta} I(0<x<\theta) + I(x \geq \theta)$$

(c) For $0 < t < \theta$, $F_{T_n}(t|\theta) = P(T_n \leq t) = P(\text{Max } X_i \leq t)$

$= P(\text{All } X_i \leq t) = P\left(\bigcap_{i=1}^n \{X_i \leq t\}\right)$

$\overset{\text{ind}}{=} \prod_{i=1}^n P(X_i \leq t) = \prod_{i=1}^n F_{X_i}(t|\theta) = \left(\frac{t}{\theta}\right)^n$, so

$$F_{T_n}(t|\theta) = \left(\frac{t}{\theta}\right)^n I(0<t<\theta) + I(t \geq \theta)$$

(d) $f_{T_n}(t|\theta) = \frac{d}{dt} F_{T_n}(t|\theta) = n\left(\frac{t}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} I(0<t<\theta)$

$$= \frac{nt^{n-1}}{\theta^n} I(0<t<\theta)$$

(e) $E(T_n) = \int_0^\theta t \, \frac{nt^{n-1}}{\theta^n}\, dt = \frac{n}{\theta^n} \int_0^\theta t^n \, dt$

$$= \frac{n}{\theta^n} \left.\frac{t^{n+1}}{n+1}\right|_0^\theta = \frac{n}{n+1} \frac{\theta^{n+1}}{\theta^n} = \frac{n}{n+1}\theta$$

Biased

(5f) $\lim_{n\to\infty} P\{|T_n-\theta|<\varepsilon\} = \lim_{n\to\infty} P\{\theta-\varepsilon<T_n<\theta+\varepsilon\}$

$= \lim_{n\to\infty} P\{\theta-\varepsilon<T_n<\theta\} = \lim_{n\to\infty}\left(1-F_{T_n}(\theta-\varepsilon)\right)$

$= 1-\lim_{n\to\infty}\left(\underbrace{\frac{\theta-\varepsilon}{\theta}}_{\text{Less than one}}\right)^n = 1-0=1$ consistent

(g) Have $\lim_{n\to\infty} E(T_n) = \lim_{n\to\infty}\frac{n}{n+1}\theta = 1\cdot\theta=\theta$

And $Var(T_n) = E(T_n^2)-(E(T_n))^2$

$E(T_n^2) = \int_0^\theta t^2\,\frac{n t^{n-1}}{\theta^n}\,dt = \frac{n}{\theta^n}\int_0^\theta t^{n+1}\,dt$

$= \frac{n}{\theta^n}\frac{t^{n+2}}{n+2}\Big|_0^\theta = \frac{n}{n+2}\theta^2$, and $Var(T_n)=E(T_n^2)-(E(T_n))^2$

$= \frac{n}{n+2}\theta^2-\left(\frac{n\theta}{n+1}\right)^2 = \theta^2\left(\frac{n}{n+2}-\frac{n^2}{(n+1)^2}\right)$ and

$\lim_{n\to\infty} Var(T_n) = \lim_{n\to\infty}\theta^2\left(\frac{n}{n+2}-\frac{n^2}{(n+1)^2}\right)$

$= \theta^2\left(\lim_{n\to\infty}\frac{n}{n+2}-\lim_{n\to\infty}\frac{n^2\,\frac{1}{n^2}}{(n+1)^2\,\frac{1}{n^2}}\right)$

$= \theta^2\left(1-\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^2}\right) = \theta^2(1-1)=0$

So $T_n \to \theta$ by the Variance rule

(5 h) $\hat{\Theta}_1 = \frac{n+1}{n} T_n$. $E(\hat{\Theta}_1) = \frac{n+1}{n} E(T_n)$

$$= \frac{n+1}{n} \cdot \frac{n}{n+1} \theta = \theta \text{ unbiased}$$

(i) $E(X_i) = E(\bar{X}_n) = \frac{\theta}{2}$, so $E(2\bar{X}_n) = 2E(\bar{X}_n)$

$= 2\frac{\theta}{2} = \theta$ unbiased

Since $\bar{X}_n \xrightarrow{P} E(X_1) = \frac{\theta}{2}$ by LLN,

$2\bar{X}_n \xrightarrow{P} 2\frac{\theta}{2} = \theta$ by continuous mapping

(j) $Var(\hat{\Theta}_1) = Var\left(\frac{n+1}{n} T_n\right) = \left(\frac{n+1}{n}\right)^2 Var(T_n)$

$$= \left(\frac{n+1}{n}\right)^2 \theta^2 \left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right)$$

$$= \theta^2 \left(\frac{(n+1)^2}{n(n+2)} - 1\right) = \theta^2 \left(\frac{\cancel{n^2} + 2n + 1 - \cancel{n^2} - \cancel{2n}}{n(n+2)}\right)$$

$$= \frac{\theta^2}{n(n+2)}$$

While $Var(\hat{\Theta}_2) = Var(2\bar{X}_n) = 4 Var(\bar{X}_n)$

$= \frac{4\sigma^2}{n}$. Looking up variance of a uniform on the formula sheet

$$= \frac{4}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n} > \frac{\theta^2}{n(n+2)} \quad \text{for } n > 1$$

$\hat{\Theta}_1$ is a <u>lot</u> better.

(6) (a) (i) $E(\hat{\beta}_1) = E\left(\frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}\right) = \frac{1}{\sum_{i=1}^n x_i^2} E\left(\sum_{i=1}^n x_i Y_i\right)$

$= \frac{1}{\sum_{i=1}^n x_i^2} \sum_{i=1}^n x_i E(Y_i) = \frac{1}{\sum_{i=1}^n x_i^2} \sum_{i=1}^n x_i E(\beta x_i + \varepsilon_i)$

$= \frac{1}{\sum_{i=1}^n x_i^2} \left(\sum_{i=1}^n (x_i^2 \beta + E(\varepsilon_i))\right) = \frac{1}{\sum_{i=1}^n x_i^2} \left(\beta \sum_{i=1}^n (x_i^2 + 0)\right)$

$= \beta \frac{\sum x_i^2}{\sum x_i^2} = \beta$ unbiased

(ii) $\mathrm{Var}(\hat{\beta}_1) = \frac{1}{\left(\sum_{i=1}^n x_i^2\right)^2} \mathrm{Var} \sum_{i=1}^n x_i Y_i$

$\overset{\text{ind}}{=} \frac{1}{(\sum x_i^2)^2} \sum_{i=1}^n x_i^2 \mathrm{Var}(\beta x_i + \varepsilon_i)$

$= \frac{1}{(\sum x_i^2)^2} \cdot \sum_{i=1}^n x_i^2 \mathrm{Var}(\varepsilon_i) = \frac{\sigma^2 \sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right)^2}$

$= \frac{\sigma^2}{\sum_{i=1}^n x_i^2} \longrightarrow 0$ as $n \to \infty$

So $\hat{\beta}_1$ is consistent by the variance rule.

(6b)(i) $E(\hat{\beta}_2) = E\left(\frac{\bar{Y}_n}{\bar{x}_n}\right) = \frac{1}{\bar{x}_n} E\left(\frac{1}{n}\sum_{i=1}^{n}(\beta x_i + \varepsilon_i)\right)$

$= \frac{1}{\bar{x}_n}\frac{1}{n}\sum_{i=1}^{n}(\beta x_i + \underbrace{E(\varepsilon_i)}_{=0}) = \frac{\beta}{\bar{x}_n}\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\beta \cancel{\bar{x}_n}}{\cancel{\bar{x}_n}}$

$= \beta$ unbiased

$Var(\hat{\beta}_2) = \frac{1}{\bar{x}_n^2}\frac{1}{n^2}\sum_{i=1}^{n} Var(x_i\beta + \varepsilon_i) = \frac{1}{n^2\bar{x}^2}\sum_{i=1}^{n}\sigma^2$

$= \frac{\cancel{n}\sigma^2}{\cancel{n}\cdot n\cdot\bar{x}_n^2}$ so $\lim_{n\to\infty} Var(\hat{\beta}_2) = \lim_{n\to\infty}\frac{\sigma^2}{n\bar{x}_n^2}$

$= \frac{\sigma^2}{c^2}\cdot 0 = 0$ & $\hat{\beta}_2$ is consistent by the variance rule!

(c) $Var(\hat{\beta}_2) \geq Var(\hat{\beta}_1) \iff \frac{\sigma^2}{n\bar{x}_n^2} \geq \frac{\sigma^2}{\sum x_i^2}$

$\iff n\bar{x}_n^2 \leq \sum_{i=1}^{n} x_i^2 \iff \sum_{i=1}^{n} x_i^2 - n\bar{x}_n^2 \geq 0$

$\iff \sum_{i=1}^{n}(x_i - \bar{x}_n)^2 \geq 0$ which is always true

Note $\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 = \sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2)$

$= \sum_{i=1}^{n} x_i^2 - 2\bar{x}_n\sum_{i=1}^{n} x_i + n\bar{x}_n^2 = \sum_{i=1}^{n} x_i^2 - 2n\bar{x}_n^2 + n\bar{x}_n^2$

$= \sum_{i=1}^{n} x_i^2 - n\bar{x}_n^2$

(7) (a) Because $E(X_i) = \frac{1}{\lambda}$, $\hat{\lambda} = \frac{1}{\overline{X}_n}$ is natural.

(b) Law of Large numbers and continuous mapping.

(c) $M_{\overline{X}_n}(t) = M_{\frac{1}{n}\sum_{i=1}^{n} X_i}(t) = M_{\sum_{i=1}^{n} X_i}\left(\frac{t}{n}\right)$

$$\stackrel{ind}{=} \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n}\left(1-\frac{t}{n\lambda}\right)^{-1} = \left(1-\frac{t}{n\lambda}\right)^{-n}$$

MGF of Gamma ($\alpha = n$, $\lambda' = n\lambda$), so

$$f_{\overline{X}_n}(x|\lambda) = \frac{(n\lambda)^n}{\Gamma(n)} e^{-n\lambda x} x^{n-1} I(x>0)$$

(d) $$E\left(\frac{1}{\overline{X}_n}\right) = \int_0^\infty \frac{1}{x} \frac{(n\lambda)^n}{\Gamma(n)} e^{-n\lambda x} x^{n-1}\, dx$$

$$= \int_0^\infty \frac{n\lambda\,(n\lambda)^{n-1}}{(n-1)\,\Gamma(n-1)} e^{-n\lambda x} x^{\overset{\text{new }\alpha}{(n-1)}-1}\, dx$$

$$= \frac{n\lambda}{n-1} \underbrace{\int_0^\infty \frac{(n\lambda)^{n-1}}{\Gamma(n-1)} e^{-n\lambda x} x^{(n-1)-1}\, dx}_{=1}$$

$= \frac{n\lambda}{n-1}$ Biased (but asymptotically unbiased)

(e) $$E\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = E\left(\frac{n-1}{n\,\overline{X}_n}\right) = \frac{n-1}{n} E\left(\frac{1}{\overline{X}_n}\right)$$

$= \frac{n-1}{n} \cdot \frac{n\lambda}{n-1} = \lambda$ unbiased

(7f) Again have $\frac{n-1}{\Sigma X_i} = \left(\frac{n-1}{n}\right) \frac{1}{\bar{X}_n}$

Because (considered as a sequence of degenerate random variables) $\frac{n-1}{n} \xrightarrow{p} 1$ & by LLN $\bar{X}_n \xrightarrow{p} \frac{1}{\lambda}$, Continuous mapping yields

$$\frac{n-1}{n} \cdot \frac{1}{\bar{X}_n} \xrightarrow{p} 1 \cdot \frac{1}{1/\lambda} = \lambda$$

consistent.

8 (a) L is unbiased iff $E(L) = \mu$

$$\iff \mu = E(L) = E\left(\sum_{i=1}^n a_i X_i\right) = \sum_{i=1}^n a_i E(X_i)$$

$$= \sum_{i=1}^n a_i \mu = \mu \sum_{i=1}^n a_i$$

$$\iff \sum_{i=1}^n a_i = 1$$

(b) $a_i = \frac{1}{n}$ for $i = 1, \ldots, n$

(c) $$Var(L) = Var\left(\sum_{i=1}^n a_i X_i\right) \overset{ind}{=} \sum_{i=1}^n a_i^2 Var(X_i)$$

$$= \sigma^2 \sum_{i=1}^n a_i^2$$ Minimize by minimizing Σa_i^2 subject to $\Sigma a_i = 1$

$$\sum_{i=1}^n a_i^2 = \sum_{i=1}^n \left(a_i - \frac{1}{n} + \frac{1}{n}\right)^2$$ coefficients for $L = \bar{X}_n$

$$= \sum_{i=1}^n \left[\left(a_i - \frac{1}{n}\right)^2 + 2\left(a_i - \frac{1}{n}\right)\cdot\frac{1}{n} + \frac{1}{n^2}\right]$$

$$= \sum_{i=1}^n \left(a_i - \frac{1}{n}\right)^2 + \frac{2}{n}\sum_{i=1}^n \left(a_i - \frac{1}{n}\right) + \sum_{i=1}^n \frac{1}{n^2}$$

$$= \sum_{i=1}^n \left(a_i - \frac{1}{n}\right)^2 + \frac{2}{n}\left(\sum_{i=1}^n a_i - \sum_{i=1}^n \frac{1}{n}\right) + \frac{n}{n^2}$$

$$= \sum_{i=1}^n \left(a_i - \frac{1}{n}\right)^2 + \frac{2}{n}(1 - 1) + \frac{1}{n}$$

$$= \sum_{i=1}^n \left(a_i - \frac{1}{n}\right)^2 + \frac{1}{n}$$

Non-negative, $= 0$ iff $a_i = \frac{1}{n}$ for $i = 1, \ldots, n$. That is, minimal when $L = \bar{X}_n$.