Assignment 1

① (a) For $x > 0$, $F_X(x) = \int_0^x 2t e^{-t^2} dt$

$u = t^2$
$du = 2t\, dt$

t	u
x	x^2
0	0

$$= \int_0^{x^2} e^{-u} du = 1 - e^{-x^2}, \text{ so}$$

$$F_X(x) = (1 - e^{-x^2}) I(x > 0)$$

(b) $P(X > \frac{1}{2}) = e^{-\frac{1}{4}} = 0.7788$

②

x	prob	x^2	$y = x^2 - 1$
−4	4/20	16	15
−3	3/20	9	8
−2	2/20	4	3
−1	1/20	1	0
0	0	0	−1
1	1/20	1	0
2	2/20	4	3
3	3/20	9	8
4	4/20	16	15

(3) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(X^2 \le y)$

$= \frac{d}{dy} P(|X| \le y^{1/2}) = \frac{d}{dy} P(-y^{1/2} \le X \le y^{1/2})$

$= \frac{d}{dy}\left(F_X(y^{1/2}) - F_X(-y^{1/2})\right)$

$= f_X(y^{1/2}) \cdot \frac{1}{2} y^{-1/2} - f_X(-y^{1/2})(-1)\frac{1}{2} y^{-\frac{1}{2}}$

$= \frac{1}{2} I(-1 < y^{1/2} < 1) \cdot \frac{1}{2} y^{-\frac{1}{2}} + \frac{1}{2} I(-1 < -y^{1/2} < 1) \cdot \frac{1}{2} y^{-\frac{1}{2}}$

$= \frac{1}{4} y^{-1/2} \left[I(0 < y^{1/2} < 1) + I(-1 < -y^{1/2} < 0) \right]$

$= \frac{1}{4} y^{-\frac{1}{2}} \left[I(0 < y < 1) + I(1 > y^{1/2} > 0) \right]$

$= \frac{1}{4} y^{-\frac{1}{2}} \left[I(0 < y < 1) + I(0 < y < 1) \right]$

$= \frac{1}{2} y^{-\frac{1}{2}} I(0 < y < 1)$

④ $f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} P(Z \le z) = \frac{d}{dz} P\left(\frac{X-\mu}{\sigma} \le z\right)$

$= \frac{d}{dz} P(X \le \sigma z + \mu) = \frac{d}{dz} F_X(\sigma z + \mu)$

$= f_X(\sigma z + \mu) \cdot \sigma = \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\sigma z + \mu - \mu)^2} \cdot \cancel{\sigma}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}$, pdf of $N(0,1)$

⑤ $M_Y(t) = E(e^{Yt}) = E\left(e^{(a + \sum_{i=1}^n b_i X_i)t}\right)$

$= E\left(e^{at} e^{\sum_{i=1}^n b_i X_i t}\right) = e^{at} E\left(e^{\sum_{i=1}^n b_i X_i t}\right)$

$= e^{at} M_{\sum_{i=1}^n b_i X_i}(t) \overset{\text{ind}}{=} e^{at} \prod_{i=1}^n M_{b_i X_i}(t)$

$= e^{at} \prod_{i=1}^n M_{X_i}(b_i t) = e^{at} \prod_{i=1}^n e^{\mu b_i t + \frac{1}{2}\sigma^2 b_i^2 t^2}$

$= e^{(a + \mu \sum_{i=1}^n b_i)t + \frac{1}{2}(\sigma^2 \sum_{i=1}^n b_i^2) t^2}$

MGF of

$N\left(a + \mu \sum_{i=1}^n b_i , \sigma^2 \sum_{i=1}^n b_i^2\right)$

(6) $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(Z^2 \le y)$

(This is zero for $y < 0$, so let $y \ge 0$)

$= \frac{d}{dy} P(|Z| \le y^{1/2}) = \frac{d}{dy} P(-y^{1/2} \le Z \le y^{1/2})$

$= \frac{d}{dy} \left(F_Z(y^{1/2}) - F_Z(-y^{1/2}) \right)$

$= f_Z(y^{1/2}) \cdot \frac{1}{2} y^{-1/2} - f_Z(-y^{\frac{1}{2}}) \cdot (-1) \cdot \frac{1}{2} y^{-\frac{1}{2}}$

$= \frac{1}{2} y^{-\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^{1/2})^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-y^{1/2})^2} \right)$

$= \frac{1}{\cancel{2}} y^{-1/2} \left(\frac{\cancel{2}}{\sqrt{2\pi}} e^{-\frac{1}{2} y} \right)$, so

$f_Y(y) = \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} e^{-\frac{1}{2} y} \, y^{\frac{1}{2} - 1} \, I(y \ge 0$

Gamma with $\alpha = \lambda = \frac{1}{2}$, which is $\chi^2(1)$.

(7) $M_Y(t) = M_{\Sigma Y_i}(t) \overset{\text{ind}}{=} \prod_{i=1}^{n} M_{Y_i}(t)$

$= \prod_{i=1}^{n} (1-2t)^{-r_i} = (1-2t)^{-\sum_{i=1}^{n} r_i}$

MGF of $\chi^2\left(\sum_{i=1}^{n} r_i\right)$

(8)

(a) $E(\bar{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{n\mu}{n} = \mu$

(b) $\text{var}(\bar{X}_n) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} X_i\right)$

$\overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

(9)

	$x=1$	$x=2$	$x=3$	
$y=1$	$3/12$ (1)	$1/12$ (2)	$3/12$ (3)	$7/12$
$y=2$	$1/12$ (2)	$3/12$ (4)	$1/12$ (6)	$5/12$
	$\frac{4}{12}$	$\frac{4}{12}$	$\frac{4}{12}$	$12/12$

(a)

$E(X) = 1\cdot\frac{4}{12} + 2\cdot\frac{4}{12} + 3\cdot\frac{4}{12} = (1+2+3)\cdot\frac{1}{3} = \frac{6}{3} = 2$

$E(Y) = 1\cdot\frac{7}{12} + 2\cdot\frac{5}{12} = \frac{17}{12}$

Put $Z = XY$ in the cells, in red

z	1	2	3	4	6
$P_Z(z)$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{1}{12}$

$E(Z) = \sum_z z\,P(z) = \frac{1}{12}(3+4+9+12+6) = \frac{34}{12}$

$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{34}{12} - 2\cdot\frac{17}{12} = 0$

(b) Not independent, because

$P_{X,Y}(1,2) = \frac{1}{12} \neq P_X(1)\,P_Y(2) = \frac{4}{12}\cdot\frac{5}{12} = \frac{20}{144}$

$\frac{1}{12} = \frac{12}{144}$

(10) $Var(X) = E\left((X-\mu)^2\right) = E(X^2 - 2\mu X + \mu^2)$

$= E(X^2) - 2\mu E(X) + E(\mu^2) = E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$

(11) $Cov(X,Y) = E\left((X-\mu_x)(Y-\mu_y)\right)$

$= E(XY - X\mu_y - \mu_x Y + \mu_x\mu_y)$

$= E(XY) - E(X)\mu_y - \mu_x E(Y) + E(\mu_x\mu_y)$

$= E(XY) - \mu_x\mu_y - \mu_x\mu_y + \mu_x\mu_y$

$= E(XY) - \mu_x\mu_y = E(XY) - E(X)E(Y)$

(12) True: $E(XY) = \sum_x \sum_y xy\, P_{XY}(x,y)$

$\overset{\text{ind}}{=} \sum_x \sum_y xy\, P_X(x) P_Y(y)$

$= \sum_x x P_X(x) \sum_y y P_Y(y) = E(X)E(Y)$

And $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

(13) False: Problem 9 is a counter-example

(14) $E(Y+Z) = E(Y) + E(Z) = \mu_y + \mu_z$, so

$$\text{cov}(X, Y+Z) = E\Big((X-\mu_x)(Y+Z-(\mu_y+\mu_z)) \Big)$$

$$= E\Big[(X-\mu_x)(Y-\mu_y + Z-\mu_z)\Big]$$

$$= E\Big[(X-\mu_x)(Y-\mu_y) + (X-\mu_x)(Z-\mu_z)\Big]$$

$$= E\Big[(X-\mu_x)(Y-\mu_y)\Big] + E\Big[(X-\mu_x)(Z-\mu_z)\Big]$$

$$= \text{Cov}(X,Y) + \text{Cov}(X,Z)$$

Could have used $\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \text{cov}(X_i, Y_j)$

(15) Impossible because $\lim_{x \to -\infty} F(x) = 0$.

Or, if X is continuous then $f_X(x) = 0$ for all x and $\int_{-\infty}^{\infty} f_X(x)\,dx = 0 \neq 1$

If X is discrete, $F_X(x)$ has no jump points, so $P_X(x) = 0$ for all x and $\sum_x P_X(x) = 0 \neq 1$

(16) $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = P(X \le x)$$

(17) $$F_{X|Y}(x|y) = P(X \le x \mid Y = y) = \frac{P(X \le x, Y = y)}{P(Y = y)}$$

$$\neq \frac{P(X \le x, Y \le y)}{P(Y \le y)} = \frac{F_{XY}(x,y)}{F_Y(y)}$$

(18) The statement is false. For example, let
$P_X(1) = P_X(2) = P_X(3) = \frac{1}{3}$. $E(X) = 2$
If $Y = \frac{1}{X}$, $P_Y(1) = P_Y(\frac{1}{2}) = P_Y(\frac{1}{3}) = \frac{1}{3}$, and
$E(Y) = E(\frac{1}{X}) = \frac{1}{3}\left(\frac{6}{6} + \frac{3}{6} + \frac{2}{6}\right) = \frac{1}{3} \cdot \frac{11}{6}$
$= \frac{11}{18} \neq \frac{1}{2}$

(19) Expected value is a linear operation. You can't just bring it inside the square.
This calculation says that every random variable has a variance of zero. All random variables are degenerate. In fact, there is no such thing as a random variable. All games of chance produce exactly the same result, every time.