Assignment 10

① (a) $L(\theta) = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} I(x_i=0,1)$

$$= \underbrace{\theta^{\Sigma x_i}(1-\theta)^{n-\Sigma x_i}}_{g(\theta,\, \Sigma x_i)} \underbrace{\prod_{i=1}^{n} I(x_i=0,1)}_{h(\underline{x})}$$

$\sum_{i=1}^{n} x_i$ is sufficient

(b) Yes, 1+1=2

② (a) $L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} I(x_i=0,1,\ldots)$

$$= \underbrace{e^{-n\lambda}\lambda^{n\bar{x}_n}}_{g(\lambda,\ }\ \underbrace{\frac{\prod_{i=1}^{n} I(x_i=0,1,\ldots)}{\prod_{i=1}^{n} x_i!}}_{h(\underline{x})}$$

$\bar{X}_n$ is sufficient

(b) $\bar{x} = (14+10+8+8)/4 = 10$

(3) (a) $L(\theta) = \prod_{i=1}^{n} \frac{2^\theta}{\Gamma(\theta)} e^{-2x_i} x_i^{\theta-1} I(x_i \geq 0)$

$$= \frac{2^{n\theta}}{\Gamma(\theta)^n} e^{-2\sum_{i=1}^{n} x_i} \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} \prod_{i=1}^{n} (x_i \geq 0)$$

$$= \underbrace{\frac{2^{n\theta}}{\Gamma(\theta)^n} \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}}_{g(\theta, t)} \underbrace{e^{-2\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} I(x_i \geq 0)}_{h(\underset{\sim}{x})}$$

$T = \prod_{i=1}^{n} X_i$ is a sufficient statistic

(b) $0.706 * 2.154 * 2.367 * 4.039 * 2.155 * 1.678$
$= 52.57288$

(4) $L(L,R) = \prod_{i=1}^{n} \frac{1}{R-L} I(L \leq x_i \leq R) = \frac{1}{(R-L)^n} \prod_{i=1}^{n} I(L \leq x_i \leq R)$

$$= \underbrace{\frac{1}{(R-L)^n} I(L \leq Min(x_i)) I(R \geq Max(x_i))}_{g((L,R), (Min(x_i), Max(x_i)))} \cdot 1$$

So $(Min(X_i), Max(X_i))$ is sufficient

(5) (a) $L(\mu, \sigma^2, \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$

$$= \frac{1}{(\sigma^2)^{n/2}(2\pi)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2}$$

$$= \frac{1}{(\sigma^2)^{n/2}(2\pi)^{n/2}} e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu)^2\right)}$$

$$= \underbrace{\frac{1}{(\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2+n(\bar{x}-\mu)^2\right)}}_{g(\theta,\, t)} \underbrace{\frac{1}{(2\pi)^{n/2}}}_{h(\underset{\sim}{x})}$$

(μ, σ^2) $(\bar{x}, s^2)$

sufficient statistic is $(\bar{X}_n, S^2)$

(b) c(mean(x), var(x))

[1] 100.160 7.468

(6) (a) $L(\theta, \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i - \delta}{\theta}} I(x_i \geq \delta)$

$= \frac{1}{\theta^n} e^{-\frac{\Sigma x_i - n\delta}{\theta}} \prod_{i=1}^{n} I(x_i \geq \delta)$

$= \underbrace{\frac{1}{\theta^n} e^{-\frac{\Sigma x_i - n\delta}{\theta}} I(\min(x_i) \geq \delta)}_{g\left((\theta,\delta),\ (\sum_{i=1}^{n} x_i,\ \min(x_i))\right)} \cdot \underbrace{1}_{h(\underset{\sim}{x})}$

Sufficient statistic is $\left(\sum_{i=1}^{n} X_i,\ \min(X_i)\right)$

b) $\text{sum}(x) = 63.65 \qquad \min(x) = 10.02$

(7) Let $Y_i = \frac{\partial}{\partial\theta} \ln P(X_i | \theta)$. $1 = \sum_x P(x|\theta)$

$\Rightarrow \frac{\partial}{\partial\theta} 1 = \frac{\partial}{\partial\theta} \sum_x P(x|\theta) = \sum_x \frac{\partial}{\partial\theta} P(x|\theta)$

$\Rightarrow 0 = \sum_x \frac{\frac{\partial}{\partial\theta} P(x|\theta)}{P(x|\theta)} \cdot P(x|\theta)$

$= \sum_x \left(\frac{\partial}{\partial\theta} \ln P(x|\theta)\right) \cdot P(x|\theta) = E(Y_i)$, and

$E\left(\ell'(\theta, \underset{\sim}{X})\right) = E\left(\frac{\partial}{\partial\theta} \ln \prod_{i=1}^{n} P(X_i|\theta)\right)$

$= E\left(\frac{\partial}{\partial\theta} \sum_{i=1}^{n} \ln P(X_i|\theta)\right) = E\left(\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln P(X_i|\theta)\right)$

$= E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} 0 = 0$

(8) (a) $\hat{\Theta}_n = \bar{X}_n$. Yes it's unbiased.

(b) $Var(\hat{\Theta}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \overset{ind}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \theta(1-\theta) = \frac{\theta(1-\theta)}{n}$$

(c) $I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2} \ln p(X|\theta)\right)$

$$= -E\left(\frac{\partial^2}{\partial\theta^2} \ln\left(\theta^X (1-\theta)^{1-X}\right)\right)$$

$$= -E\left(\frac{\partial^2}{\partial\theta^2}\left(X \ln\theta + (1-X)\ln(1-\theta)\right)\right)$$

$$= -E\left(\frac{\partial}{\partial\theta}\left(\frac{X}{\theta} + \frac{(1-X)}{1-\theta}(-1)\right)\right)$$

$$= -E\left(\frac{\partial}{\partial\theta}\left(X\theta^{-1} - (1-X)(1-\theta)^{-1}\right)\right)$$

$$= -E\left(-\frac{X}{\theta^2} - (1-X)(-1)(1-\theta)^{-2}(-1)\right)$$

$$= -E\left(\frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right) = \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2}$$

$= \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1-\theta+\theta}{\theta(1-\theta)} = \frac{1}{\theta(1-\theta)}$, and the

C-R lower bound is $\frac{1}{n I(\theta)} = \frac{1}{n \frac{1}{\theta(1-\theta)}} = \frac{\theta(1-\theta)}{n}$

(d) $= Var(\hat{\Theta}_n)$ Yes

9 (a) $E(\hat{\Theta}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2)$

$= \frac{1}{n}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n}\sum_{i=1}^{n} \theta = \frac{n\theta}{n} = \theta$ Yes unbiased

(b) $\frac{1}{\theta}\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n}\left(\frac{X_i - 0}{\sqrt{\theta}}\right)^2$ sum of indep. $\chi^2(1)$s : $\chi^2(n)$

From Formula sheet, $Var = 2r = 2n$, so

$Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2} Var\left(\theta \frac{1}{\theta}\sum_{i=1}^{n} X_i^2\right) = \frac{\theta^2}{n^2} Var\left(\frac{1}{\theta}\sum_{i=1}^{n} X_i^2\right)$

$= \frac{\theta^2}{n^2}\cdot 2n = \frac{2\theta^2}{n}$

(c) $I(\theta) = -E\frac{\partial^2}{\partial\theta^2}\ln f(x|\theta)$

$= -E\frac{\partial^2}{\partial\theta^2}\ln\left(\frac{1}{\theta^{1/2}\sqrt{2\pi}} e^{-\frac{1}{2\theta}X^2}\right)$

$= -E\frac{\partial^2}{\partial\theta^2}\ln\left(\theta^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2\theta}X^2}\right)$

$= -E\frac{\partial^2}{\partial\theta^2}\left(-\frac{1}{2}\ln\theta - \frac{1}{2}\ln 2\pi - \frac{1}{2}X^2\theta^{-1}\right)$

$= -E\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\frac{1}{\theta} - 0 - \frac{X^2}{2}(-1)\theta^{-2}\right.$

(9 c continued)

$$= -E\,\frac{\partial}{\partial\theta}\left(-\frac{1}{2}\theta^{-1} + \frac{X^2}{2}\theta^{-2}\right)$$

$$= -E\left(-\frac{1}{2}(-1)\theta^{-2} + \frac{X^2}{\cancel{2}}(-\cancel{2})\theta^{-3}\right)$$

$$= -E\left(\frac{1}{2\theta^2} - \frac{X^2}{\theta^3}\right) = \frac{E(X^2)}{\theta^3} - \frac{1}{2\theta^2}$$

$$= \frac{\theta}{\theta^3} - \frac{1}{2\theta^2} = \frac{1}{\theta^2} - \frac{1}{2\theta^2} = \frac{1}{2\theta^2}$$

And Cramér-Rao lower bound is

$$\frac{1}{n I(\theta)} = \frac{1}{n\,\frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

(d) Yes: $Var(\hat{\Theta}_n) = \frac{2\theta^2}{n}$ from part (b).

10 (a) $l'(\theta) = \frac{\partial}{\partial\theta} \ln \prod_{i=1}^{n} (1-\theta)^{x_i}\theta = \frac{\partial}{\partial\theta} \ln\left((1-\theta)^{n\bar{x}} \theta^n\right)$

$= \frac{\partial}{\partial\theta}\left(n\bar{x} \ln(1-\theta) + n \ln\theta\right) = \frac{-n\bar{x}}{1-\theta} + \frac{n}{\theta} \overset{\text{set}}{=} 0$

$\Longrightarrow \frac{\cancel{n}\bar{x}}{1-\theta} = \frac{\cancel{n}}{\theta} \Longrightarrow \theta\bar{x} = 1-\theta$

$\Longrightarrow \theta(1+\bar{x}) = 1 \Longrightarrow \hat{\theta} = \frac{1}{1+\bar{X}_n}$

(b) $\hat{\theta} = \frac{1}{1+0.85} = 0.54$. For the CI, need $I(\theta)$

$I(\theta) = -E\frac{\partial^2}{\partial\theta^2} \ln\left((1-\theta)^X\theta\right)$

$= -E\frac{\partial^2}{\partial\theta^2}\left(X\ln(1-\theta) + \ln\theta\right)$

$= -E\frac{\partial}{\partial\theta}\left(\frac{-X}{1-\theta} + \frac{1}{\theta}\right)$

$= -E\frac{\partial}{\partial\theta}\left(-X(1-\theta)^{-1} + \theta^{-1}\right)$

$= -E\left(-X(-1)(1-\theta)^{-2}(-1) - \theta^{-2}\right)$

$= -E\left(\frac{-X}{(1-\theta)^2} - \frac{1}{\theta^2}\right) = E\left(\frac{X}{(1-\theta)^2} + \frac{1}{\theta^2}\right)$

(10b cont.)

$$= \frac{E(X)}{(1-\theta)^2} + \frac{1}{\theta^2} = \frac{(1-\theta)/\theta}{(1-\theta)^2} + \frac{1}{\theta^2}$$

$$= \frac{1}{\theta(1-\theta)} + \frac{1}{\theta^2} = \frac{\theta}{\theta^2(1-\theta)} + \frac{1-\theta}{\theta^2(1-\theta)}$$

$$= \frac{1}{\theta^2(1-\theta)}$$, and using the 2nd formula for the CLT for MLEs, CI is

$$\hat{\Theta}_n \pm z_{1-\alpha/2} \frac{\sqrt{1/I(\hat{\Theta}_n)}}{\sqrt{n}} = \hat{\Theta}_n \pm z_{1-\alpha/2} \frac{\sqrt{\hat{\Theta}^2(1-\hat{\Theta})}}{\sqrt{n}}$$

$$= 0.54 \pm 1.96 \sqrt{\frac{0.54^2(1-0.54)}{100}} = 0.54 \pm 0.0718$$

$$= (0.4682, 0.6118)$$

11 (a) CI is $\hat{\Theta}_n \pm z_{1-\alpha/2} \dfrac{\sqrt{1/I(\hat{\Theta}_n)}}{\sqrt{n}}$

From Q 9, $I(\theta)$ is $\frac{1}{2\theta^2}$, so CI is

$$\hat{\Theta}_n \pm z_{1-\alpha/2} \sqrt{\frac{2\hat{\Theta}_n^2}{n}} \quad \hat{\Theta}_n = \frac{\sum x_i^2}{n} = \frac{480.38}{120}$$

yielding

$$= 4.003$$

$$4.003 \pm 1.96 \sqrt{\frac{2 * 4.003^2}{120}} = 4.003 \pm 1.013$$

$= (2.99, 5.016)$

b) $\frac{1}{\theta}\sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$, so

$$1-\alpha = P\left(\chi^2_{\alpha/2}(n) \le \frac{1}{\theta}\sum_{i=1}^{n} X_i \le \chi^2_{1-\alpha/2}(n)\right)$$

$$= P\left(\frac{1}{\chi^2_{\alpha/2}(n)} \ge \frac{\theta}{\sum_{i=1}^{n} X_i^2} \ge \frac{1}{\chi^2_{1-\alpha/2}(n)}\right)$$

$$= P\left(\frac{\sum_{i=1}^{n} X_i^2}{\chi^2_{1-\alpha/2}(n)} \le \theta \le \frac{\sum_{i=1}^{n} X_i^2}{\chi^2_{\alpha/2}(n)}\right)$$

qchisq (0.025, 120) = 91.57, qchisq (0.975, 120) = 152.21

$$= \left(\frac{480.38}{152.21}, \frac{480.38}{91.57}\right) = (3.16, 5.25)$$

(12) (a) $P_\theta\left(Y \geq \chi^2_{1-\alpha}(n)\right) = P_\theta\left(\frac{1}{\theta_0}\sum X_i^2 \geq \chi^2_{1-\alpha}(n)\right)$

$$= P_\theta\left(\frac{\cancel{\theta_0}}{\theta}\frac{1}{\cancel{\theta_0}}\sum_{i=1}^{n} X_i^2 \geq \frac{\theta_0}{\theta}\chi^2_{1-\alpha}(n)\right)$$

$$= P_\theta\left(\frac{1}{\theta}\sum_{i=1}^{n} X_i^2 \geq \underbrace{\frac{\theta_0}{\theta}\chi^2_{1-\alpha}(n)}_{\frac{4}{4.25}(146.57)}\right)$$

$$= 1 - \text{pchisq}(137.95, 120) = 0.125$$

b) $P_\theta\left(\frac{\sqrt{n}(\hat{\Theta}_n - \theta_0)}{\sqrt{1/I(\theta_0)}} \geq z_{1-\alpha}\right)$

$$= P_\theta\left(\frac{\sqrt{n}(\hat{\Theta}_n - \theta_0)}{\sqrt{2\theta_0^2}} \geq z_{1-\alpha}\right) = P_\theta\left(\frac{\sqrt{n}(\hat{\Theta}_n - \theta_0)}{\theta_0\sqrt{2}} \geq z_{1-\alpha}\right)$$

$$= P_\theta\left(\hat{\Theta}_n - \theta_0 \geq \frac{\theta_0\sqrt{2}}{\sqrt{n}} z_{1-\alpha/2}\right)$$

(12b continued)

$$= P_\theta\left(\hat{\Theta}_n \geq \theta_0 + \frac{\theta_0\sqrt{2}}{\sqrt{n}} z_{1-\alpha}\right)$$

$$= P_\theta\left(\hat{\Theta}_n - \theta \geq \theta_0 - \theta + \frac{\theta_0\sqrt{2}}{\sqrt{n}} z_{1-\alpha}\right)$$

$$= P_\theta\left(\sqrt{n}(\hat{\Theta}_n - \theta) \geq \sqrt{n}(\theta_0 - \theta) + \cancel{\sqrt{n}}\frac{\theta_0\sqrt{2}}{\cancel{\sqrt{n}}} z_{1-\alpha}\right)$$

$$= P_\theta\left(\frac{\sqrt{n}(\hat{\Theta}_n - \theta)}{\theta\sqrt{2}} \geq \frac{\sqrt{n}(\theta_0 - \theta)}{\theta\sqrt{2}} + \frac{\theta_0\cancel{\sqrt{2}}}{\theta\cancel{\sqrt{2}}} z_{1-\alpha}\right)$$

$$\xrightarrow{n\to\infty} 1 - \Phi\left(\frac{\sqrt{n}(\theta_0 - \theta)}{\theta\sqrt{2}} + \frac{\theta_0}{\theta} z_{1-\alpha}\right)$$

$$\approx 1 - \Phi\left(\frac{\sqrt{120}(4 - 4.25)}{4.25\sqrt{2}} + \frac{4}{4.25}(1.645)\right)$$

$$= 1 - \text{pnorm}(1.0926) = 1 - 0.863$$

$$= 0.138$$

compare 0.125 from the exact test
BOTH ARE VERY LOW.