Joint Distributions: Part Two[1]

STA 256: Fall 2018

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Overview

1. Independence
2. Conditional Distributions
3. Transformations

Independent Random Variables
Discrete or Continuous

The random variables X and Y are said to be *independent* if

$$F_{xy}(x, y) = F_x(x)F_y(y)$$

For all real x and y.

Theorem (for discrete random variables)

Recalling independence means $F_{xy}(x,y) = F_x(x)F_y(y)$

The discrete random variables X and Y are independent if and only if

$$p_{xy}(x,y) = p_x(x)\,p_y(y)$$

for all real x and y.

Theorem (for continuous random variables)

Recalling independence means $F_{xy}(x, y) = F_x(x)F_y(y)$

The continuous random variables X and Y are independent if and only if

$$f_{xy}(x, y) = f_x(x)\, f_y(y)$$

for all real x and y.

Conditional Distributions

Of discrete random variables

If X and Y are discrete random variables, the conditional probability mass function of X given $Y = y$ is just a conditional probability. It is given by

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

These are just probabilities of events. For example,

$$P(X = x, Y = y) = P\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}$$

We write

$$p_{x|y}(x|y) = \frac{p_{x,y}(x, y)}{p_y(y)}$$

Note that $p_{x|y}(x|y)$ is defined only for y values such that $p_y(y) > 0$.

Conditional Probability Mass Functions

Both ways

$$p_{y|x}(y|x) = \frac{p_{x,y}(x,y)}{p_x(x)}$$

$$p_{x|y}(x|y) = \frac{p_{x,y}(x,y)}{p_y(y)}$$

Defined where the denominators are non-zero.

Independence makes sense

In terms of conditional probability mass functions

Suppose X and Y are independent. Then $p_{xy}(x,y) = p_x(x)p_y(y)$, and

$$\begin{aligned} p_{x|y}(x|y) &= \frac{p_{x,y}(x,y)}{p_y(y)} \\ &= \frac{p_x(x)p_y(y)}{p_y(y)} \\ &= p_x(x) \end{aligned}$$

So we see that the conditional distribution of X given $Y = y$ is identical for every value of y. It does not depend on the value of y.

The other way

Suppose the conditional distribution of X given $Y = y$ does not depend on the value of y. Then

$$
\begin{aligned}
& p_{x|y}(x|y) = p_x(x) \\
\Leftrightarrow \quad & p_x(x) = \frac{p_{x,y}(x,y)}{p_y(y)} \\
\Leftrightarrow \quad & p_{x,y}(x,y) = p_x(x)\, p_y(y)
\end{aligned}
$$

So that X and Y are independent.

Conditional distributions of continuous random variables

If X and Y are continuous random variables, the conditional probability density of X given $Y = y$ is

$$f_{x|y}(x|y) = \frac{f_{x,y}(x, y)}{f_y(y)}$$

- Note that $f_{x|y}(x|y)$ is defined only for y values such that $f_y(y) > 0$.
- It looks like we are conditioning on an event of probability zero, but the conditional density is a limit of a conditional probability, as the radius of a tiny region surrounding (x, y) goes to zero.

Conditional Probability Density Functions

Both ways

$$f_{y|x}(y|x) = \frac{f_{x,y}(x,y)}{f_x(x)}$$

$$f_{x|y}(x|y) = \frac{f_{x,y}(x,y)}{f_y(y)}$$

Defined where the denominators are non-zero.

Independence makes sense

In terms of conditional densities

Suppose X and Y are independent. Then $f_{xy}(x,y) = f_x(x)f_y(y)$, and

$$\begin{aligned} f_{x|y}(x|y) &= \frac{f_{x,y}(x,y)}{f_y(y)} \\ &= \frac{f_x(x)f_y(y)}{f_y(y)} \\ &= f_x(x) \end{aligned}$$

And we see that the conditional density of X given $Y = y$ is identical for every value of y. It does not depend on the value of y.

The other way

Suppose the conditional density of X given $Y = y$ does not depend on the value of y. Then

$$\begin{aligned} & f_{x|y}(x|y) = f_x(x) \\ \Leftrightarrow \quad & f_x(x) = \frac{f_{x,y}(x,y)}{f_y(y)} \\ \Leftrightarrow \quad & f_{x,y}(x,y) = f_x(x)\, f_y(y) \end{aligned}$$

So that X and Y are independent.

Transformations of Jointly Distributed Random Variables

Let $Y = g(X_1, \ldots, X_n)$. What is the probability distribution of Y?

For example,

- X_1 is the number of jobs completed by employee 1.
- X_2 is the number of jobs completed by employee 2.
- You know the probability distributions of X_1 and X_2.
- You would like to know the probability distribution of $Y = X_1 + X_2$.

Convolutions of discrete random variables

- Let X and Y be discrete random variables.
- The standard case is where they are independent.
- Want probability mass function of $Z = X + Y$.

$$\begin{aligned}
p_z(z) &= P(Z = z) \\
&= P(X + Y = z) \\
&= \sum_x P(X + Y = z | X = x) P(X = x) \\
&= \sum_x P(x + Y = z | X = x) P(X = x) \\
&= \sum_x P(Y = z - x | X = x) P(X = x) \\
&= \sum_x P(Y = z - x) P(X = x) \text{ by independence} \\
&= \sum_x p_x(x) p_y(z - x)
\end{aligned}$$

Summarizing

Convolutions of discrete random variables

Let X and Y be *independent* discrete random variables, and

$Z = X + Y$.

$$p_z(z) = \sum_x p_x(x) p_y(z - x)$$

Two Important results

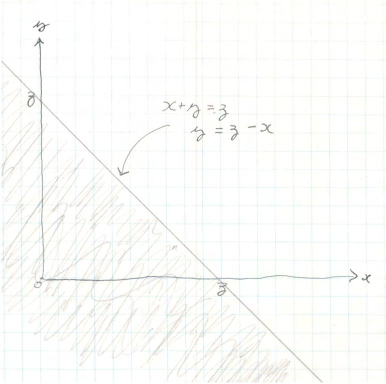
Proved using the convolution formula

- Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent. Then $Z = X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$.
- Let $X \sim \text{Binomial}(n_1, p)$ and $Y \sim \text{Binomial}(n_2, p)$ be independent. Then $Z = X + Y \sim \text{Binomial}(n_1 + n_2, p)$

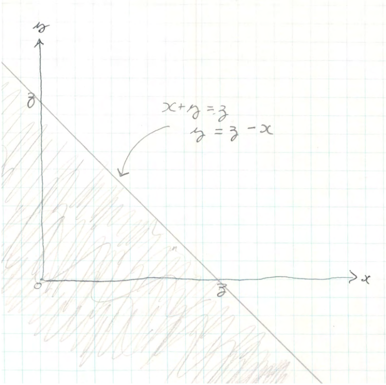
Convolutions of *continuous* random variables

- Let X and Y be continuous random variables.
- The standard case is where they are independent.
- Want probability density function of $Z = X + Y$.

$$\begin{aligned} f_z(z) &= \frac{d}{dz} P(Z \leq z) \\ &= \frac{d}{dz} P(X + Y \leq z) \end{aligned}$$



Continuing . . .



$$
\begin{aligned}
f_z(z) &= \frac{d}{dz} P(X+Y \leq z) \\
&= \frac{d}{dz} \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{xy}(x,y)\, dy\, dx
\end{aligned}
$$

$t = y + x \quad y = t - x \quad dy = dt$

y	$t = y + x$
$z - x$	z
$-\infty$	$-\infty$

$\int_{-\infty}^{z} f_{xy}(x, t-x)\, dt$

Still continuing, have

$$
\begin{aligned}
f_z(z) &= \frac{d}{dz}\int_{-\infty}^{\infty}\int_{-\infty}^{z} f_{xy}(x, t-x)\, dt\, dx \\
&= \frac{d}{dz}\int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{xy}(x, t-x)\, dx\, dt \\
&= \int_{-\infty}^{\infty} f_{xy}(x, z-x)\, dx \\
&= \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\, dx \quad \text{if } X \text{ and } Y \text{ are independent.}
\end{aligned}
$$

Compare

For continuous random variables:

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\, dx$$

For discrete random variables:

$$p_z(z) = \sum_x p_x(x) p_y(z-x)$$

Of course you need to pay attention to the limits of integration or summation, because $f_x(x)f_y(z-x)$ may be zero for some x.

Two Important results for continuous random variables

Proved using the convolution formula

- Let X and Y be independent exponential random variables with parameter $\lambda > 0$. Then $Z = X + Y \sim \text{Gamma}(\alpha = 2, \lambda)$.
- Let $X \sim \text{Normal}(\mu_1, \sigma_1)$ and $Y \sim \text{Normal}(\mu_2, \sigma_2)$ be independent. Then $Z = X + Y \sim \text{Normal}\left(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2}\right)$.

The Jacobian Method

- X_1 and X_2 are continuous random variables.
- $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$.
- Want $f_{y_1 y_2}(y_1, y_2)$

Solve for x_1 and x_2, obtaining $x_1(y_1, y_2)$ and $x_2(y_1, y_2)$. Then

$$f_{y_1 y_2}(y_1, y_2) = f_{x_1 x_2}(\, x_1(y_1, y_2), x_2(y_1, y_2)\,) \cdot abs \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

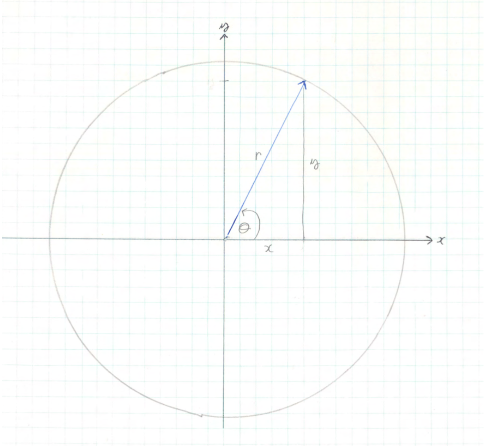
More about the Jacobian method

$Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$

- It follows directly from a change of variables formula in multi-variable integration. The proof is omitted.
- It must be possible to solve $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ for x_1 and x_2.
- That is, the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ must be one to one (injective).
- Frequently you are only interested in Y_1, and $Y_2 = g_2(X_1, X_2)$ is chosen to make reverse solution easy.
- The partial derivatives must all be continuous, except possibly on a set of probability zero (they almost always are).
- It extends naturally to higher dimension.

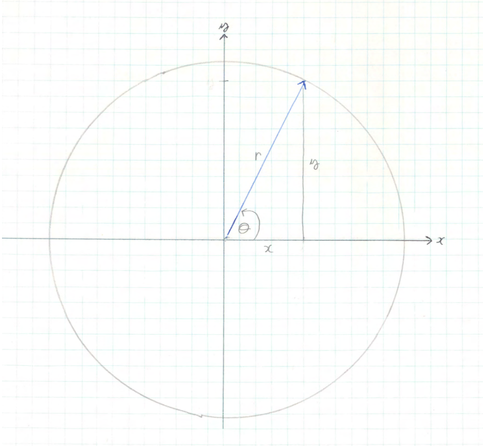
Change from rectangular to polar co-ordinates
By the Jacobian method

A point on the plane may be represented as (x, y), or



An angle θ and a radius r.

Change of variables



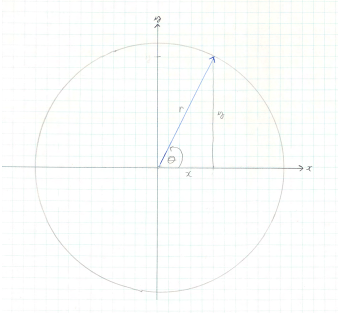
$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$
$$x^2 + y^2 = r^2$$

- As x and y range from $-\infty$ to ∞,
- r goes from 0 to ∞
- And θ goes from θ to 2π.

Integral $\int_0^\infty \int_0^\infty f_{x,y}(x,y)\,dx\,dy$

Change of variables:

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$



$$\int_0^\infty \int_0^\infty f_{x,y}(x,y)\,dx\,dy$$

$$= \int_0^{\pi/2} \int_0^\infty f_{x,y}(r\cos\theta, r\sin\theta)\, abs \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

Evaluate the determinant
(with $x = r\cos(\theta)$ and $y = r\sin(\theta)$)

$$\begin{aligned}
\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} &= \begin{vmatrix} \frac{\partial\, r\cos(\theta)}{\partial r} & \frac{\partial\, r\cos(\theta)}{\partial \theta} \\ \frac{\partial\, r\sin(\theta)}{\partial r} & \frac{\partial\, r\sin(\theta)}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} \\
&= r\cos^2\theta - -r\sin^2\theta \\
&= r(\sin^2\theta + \cos^2\theta) \\
&= r
\end{aligned}$$

So the integral is

$$\int_0^\infty \int_0^\infty f_{x,y}(x,y)\,dx\,dy = \int_0^{\pi/2} \int_0^\infty f_{x,y}(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

- The standard formula for change from rectangular to polar co-ordinates is $dx\,dy = r\,dr\,d\theta$.
- It comes from a Jacobian.
- Other limits of integration are possible.

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