Sample Questions: Joint Distributions Part 1

STA256 Fall 2018. Copyright information is at the end of the last page.

---

1. The discrete random variables x and y have joint probability mass function $p_{xy} = cxy$ for $x = 1, 2, 3$, $y = 1, 2$, and zero otherwise.

   (a) Find the value of the constant c and calculate the marginal frequency functions.

$y \backslash x$	1	2	3	
1	1/18	2/18	3/18	6/18
2	2/18	4/18	6/18	12/18
	3/18	6/18	9/18	

$$c = \frac{1}{18}$$

   (b) What is $F_x(x)$?

$$F_X(x) = P(X \le x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{6} = P(X=1) & \text{for } 1 \le x < 2 \\ \frac{1}{2} = P(X=1) + P(X=2) & \text{for } 2 \le x < 3 \\ 1 = P(X=1) + P(X=2) + P(X=3) & \text{for } x \ge 3 \end{cases}$$

6

2. The discrete random variables x and y have joint distribution

	$x=1$	$x=2$	$x=3$
$y=1$	3/12	1/12	3/12
$y=2$	1/12	3/12	1/12

Give the following. The answers are numbers.

(a) $F_{xy}(1,1) = \frac{3}{12}$ $\qquad$ $F_{xy}(2,2) = \frac{8}{12}$

(b) $F_{xy}(1.5,4) = \frac{4}{12}$ $\qquad$ $F_{xy}(-1,3) = P(X \leq -1 \cap Y \leq 3) = 0$

(c) $F_{xy}(4,4) = 1$ $\qquad$ $F_{xy}(6,1.82) = 7/12$

(d) $F_{xy}(4,19) = 1$ $\qquad$ $F_{xy}(0,0) = 0$

	$x=1$	$x=2$	$x=3$
$y=1$	3/12	1/12	3/12
$y=2$	1/12	3/12	1/12

3. A jar contains 30 red marbles, 50 green marbles and 20 blue marbles. A sample of 15 marbles is selected *with replacement.* Let X be the number of red marbles and Y be the number of blue marbles. What is the joint probability mass function of X and Y?

$$p(x, y) = \begin{cases} \binom{15}{x \; y \; 15-x-y} .3^x .2^y .5^{15-x-y} & \text{for } x, y \text{ integers}, x \geq 0, y \geq 0, x+y \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

4. This time the selection is without replacement. Again, what is the joint probability mass function of X and Y?

$$p(x, y) = \begin{cases} \dfrac{\binom{30}{x}\binom{20}{y}\binom{50}{15-x-y}}{\binom{100}{15}} & \text{for } x, y \text{ integers } x \geq 0, y \geq 0, x+y \leq 15 \\ 0 & \text{otherwise} \end{cases}$$

5. Let $f_{x,y}(x,y) = \begin{cases} c(x+y) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$

(a) Find the constant c.

$$1 = \int_0^1 \int_0^x c(x+y)\,dy\,dx$$

$$= c\int_0^1 \left(\int_0^x x\,dy + \int_0^x y\,dy \right) dx$$

$$= c\int_0^1 \left(xy\Big|_0^x + \frac{y^2}{2}\Big|_0^x \right) dx = c\int_0^1 \left(x^2 + \frac{x^2}{2}\right) dx$$

$$= c\int_0^1 \frac{3}{2}x^2\,dx = \frac{3c}{2}\int_0^1 x^2\,dx = \frac{\cancel{3}c}{2}\,\frac{x^3}{\cancel{3}}\Big|_0^1$$

$$= c/2 = 1 \text{ so } c = 2$$

(b) What is $f_x(x)$?

For $0 \le x \le 1$

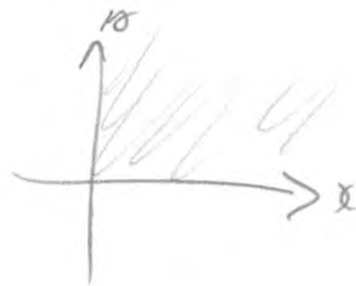
$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \cancel{\int_{-\infty}^{0} f(x,y)\,dy} + \int_0^x f(x,y)\,dy + \cancel{\int_x^{\infty} f(x,y)\,dy}$$

$$= \int_0^x 2(x+y)\,dy$$

$$= 2\left(\int_0^x x\,dy + \int_0^x y\,dy \right) = 2\left(xy\Big|_0^x + \frac{y^2}{2}\Big|_0^x \right)$$

$$= 2\left(x^2 + \frac{x^2}{2}\right) = \cancel{2}\,\frac{3}{\cancel{2}}\,x^2 \text{ so } f_x(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(5.5) $f_{XY}(x,y) = \begin{cases} \dfrac{6}{(x+1)^3(y+1)^4} & \text{for } \begin{array}{l} x \geq 0 \\ y \geq 0 \end{array} \\ 0 & \text{otherwise} \end{cases}$



Find $F_{XY}(x,y)$

For $x > 0$ and $y > 0$

$$\int_0^x \int_0^y \frac{6}{(s+1)^3(t+1)^4}\,dt\,ds = 6\int_0^x \frac{1}{(s+1)^3}\left(\int_0^y \frac{1}{(t+1)^4}\,dt\right)ds$$

$$= 6\int_0^x \frac{1}{(s+1)^3}\int_1^{y+1} u^{-4}\,du\,ds \qquad u = t+1 \quad du = dt \quad \begin{array}{c|c} t & u = t+1 \\ \hline y & y+1 \\ 0 & 1 \end{array}$$

$$= 6\int_0^x \frac{1}{(s+1)^3} \left.\frac{u^{-3}}{-3}\right|_1^{y+1} ds$$

$$= 2\int_0^x \frac{1}{(s+1)^3}\left(\frac{1}{(y+1)^3} - \frac{1}{1^3}\right)(-1)\,ds = 2\left(1 - \frac{1}{(y+1)^3}\right)\int_0^x \frac{1}{(s+1)^3}\,ds$$

$$u = s+1 \quad du = ds \quad \begin{array}{c|c} s & u \\ \hline x & x+1 \\ 0 & 1 \end{array}$$

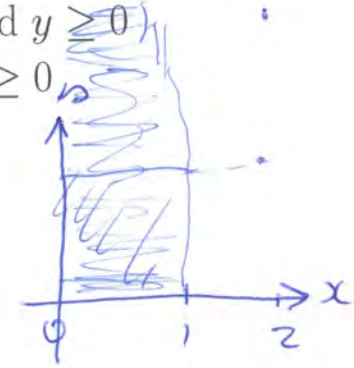
$$= 2\left(1 - \frac{1}{(y+1)^3}\right)\int_1^{x+1} u^{-3}\,du$$

$$= \cancel{2}\left(1 - \frac{1}{(y+1)^3}\right) \left.\frac{u^{-2}}{-\cancel{2}}\right|_1^{x+1} = \left(1 - \frac{1}{(y+1)^3}\right)\left(1 - \frac{1}{(x+1)^2}\right)$$

$$F_{XY}(x,y) = \begin{cases} \left(1 - \dfrac{1}{(y+1)^3}\right)\left(1 - \dfrac{1}{(x+1)^2}\right) & \text{for } \begin{array}{l} x \geq 0 \\ y \geq 0 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

6. The continuous random variables X and Y have joint cumulative distribution function

$$F_{xy}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



(a) What is $F_{xy}(\frac{1}{2}, 3)$?

$$\frac{1}{8} - \frac{1}{8}e^{-3/4} = 0.0695$$

(b) What is $F_{xy}(2, 3)$?

$$1 - e^{-y/4} = 1 - e^{-3/4} = 0.5276$$

(c) What is $F_{xy}(-1, 3)$? $= 0$

(d) What is $f_{xy}(x, y)$? For $0 \le x \le 1$ and $y \ge 0$

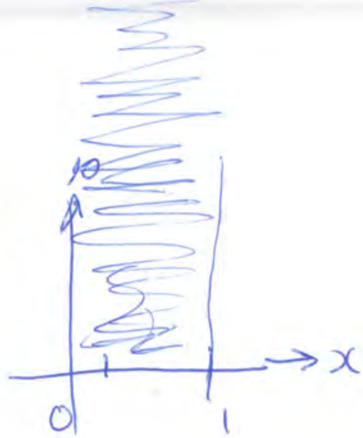
$$f_{xy}(x,y) = \frac{\partial^2 F}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(3x^2 - 3x^2 e^{-y/4}\right)$$

$$= 0 - 3x^2 e^{-y/4}\left(-\frac{1}{4}\right) = \frac{1}{4} e^{-y/4} 3x^2$$

For $x > 1$ and $y \ge 0$

$$f_{xy}(x,y) = \frac{\partial^2 F}{\partial y \, \partial x} = \frac{\partial}{\partial y} \underbrace{\frac{\partial}{\partial x}\left(1 - e^{-y/4}\right)}_{=0} = 0$$

$$f_{xy}(x,y) = \begin{cases} 3x^2 \frac{1}{4} e^{-y/4} & \text{for } 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



7. Still for the joint distribution with

$$F_{xy}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$f_{x,y}(x,y) = \begin{cases} 3x^2 \frac{1}{4} e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

For $0 \le x \le 1$ (a) Obtain $f_x(x)$ by integrating out y. $f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$

$$= \int_0^\infty 3x^2 \frac{1}{4} e^{-y/4}\,dy = 3x^2 \underbrace{\int_0^\infty \frac{1}{4} e^{-y/4}\,dy}_{\lambda e^{-\lambda y} = 1}$$

So

$$f_x(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Calculate $F_x(x)$ by taking limits. For $0 \le x \le 1$

$$F_x(x) = \lim_{y \to \infty} F_{xy}(x,y) = \lim_{y \to \infty} x^3(1 - e^{-y/4})$$

$$= x^3 \left(1 - \lim_{y \to \infty} \frac{1}{e^{y/4}}\right) = x^3$$

For $x > 1$ $\lim_{y \to \infty} (1 - e^{-y/4}) = 1 - 0 = 1$

~~(c) Obtain $f_x(x)$ from $F_x(x)$.~~ So $\cancel{F_x(x) = \begin{cases} x^3 & \text{for } 0 \le x \le \\ 0 \end{cases}}$

$$F_x(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

Show it = 1 without recognizing exponential

$$\underbrace{\int_0^\infty \boxed{\frac{1}{4}}\, e^{-y/4}\, dy}$$

$u = -\frac{1}{4} y$

$du = -\frac{1}{4} dy$

y	$u = -\frac{1}{4} y$
∞	$-\infty$
0	0

$$= -\int_0^{-\infty} e^u \, du$$

$$= (-1)\, e^u \Big|_0^{-\infty}$$

$$= (-1)\left(\lim_{u \to -\infty} e^u - 1\right)$$

$$= (-1)(0 - 1) = 1$$

c) $F_Y(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^3 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$

$f_X(x) = 0$ for $x < 0$ & $x > 1$

for $0 \le x \le 1$ $\quad \frac{d}{dx} x^3 = 3x^2$

so

$$f_Y(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

For $F_{xy}(x,y) = \begin{cases} x^3 - x^3 e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 1 - e^{-y/4} & \text{for } x > 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$ and

$$f_{x,y}(x,y) = \begin{cases} 3x^2 \frac{1}{4} e^{-y/4} & \text{for } 0 \le x \le 1 \text{ and } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) Obtain $f_y(y)$ by integrating out x.

For $y > 0$, $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

$= \int_0^1 3x^2 \frac{1}{4} e^{-y/4} dx$

$= \frac{1}{4} e^{-y/4} \int_0^1 3x^2 dx = \frac{1}{4} e^{-y/4} \cancel{3} \frac{x^3}{\cancel{3}} \Big|_0^1 = \frac{1}{4} e^{-y/4}$

$$f_Y(y) = \begin{cases} \frac{1}{4} e^{-y/4} & y \ge 0 \\ 0 & y < 0 \end{cases} \quad ✓$$

(e) Obtain $F_y(y)$ by taking limits.

For $y > 0$ $F_Y(y) = \lim_{x \to \infty} F_{XY}(x,y)$ After a while, $x > 1$

$= \lim_{x \to \infty} (1 - e^{-y/4}) = 1 - e^{-y/4}$, so

$$F_Y(y) = \begin{cases} 1 - e^{-y/4} & y \ge 0 \\ 0 & y < 0 \end{cases}$$

(f) Obtain $f_y(y)$ from Obtain $F_y(y)$.

For $y \ge 0$,

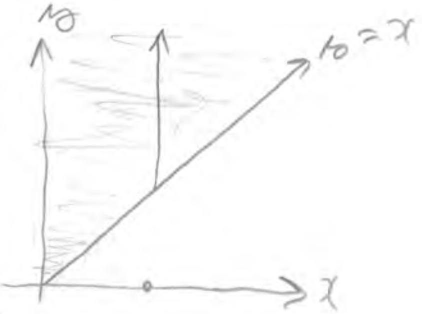
$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(1 - e^{-y/4})$

$= -e^{-y/4}\left(-\frac{1}{4}\right) = \frac{1}{4} e^{-y/4}$, and

$$f_Y(y) = \begin{cases} \frac{1}{4} e^{-y/4} & \text{for } y \ge 0 \\ 0 & \text{for } y < 0 \end{cases}$$

8. Let $f_{x,y}(x,y) = \begin{cases} 2e^{-(x+y)} & \text{for } 0 \leq x \leq y \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Obtain $f_x(x)$.



For fixed $x > 0$

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{\infty} 2e^{-x}e^{-y}\,dy$$

$u = -y \quad du = -dy$

y	$u=-y$
∞	$-\infty$
x	$-x$

$$= 2e^{-x}(-1)\int_x^{\infty} e^{-y}\,(-dy)$$

$$= -2e^{-x}\, e^{u}\Big|_{-x}^{-\infty} = -2e^{-x}(0 - e^{-x})$$

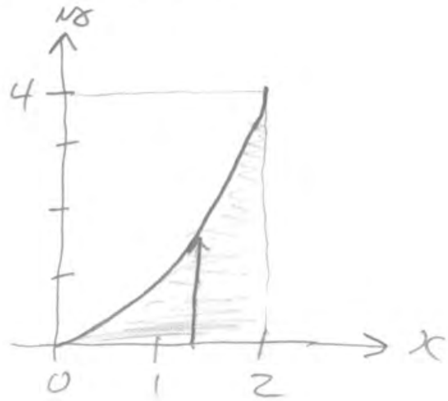
$$= 2e^{-2x}$$

, so

$$f_x(x) = \begin{cases} 2e^{-2x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

9. Let $f_{x,y}(x,y) = \begin{cases} \frac{xy}{16} & \text{for } 0 \le x \le 2 \text{ and } 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$

Find $P(Y < X^2)$. The answer is a number.



$$\int_0^2 \int_0^{x^2} \frac{xy}{16} \, dy \, dx$$

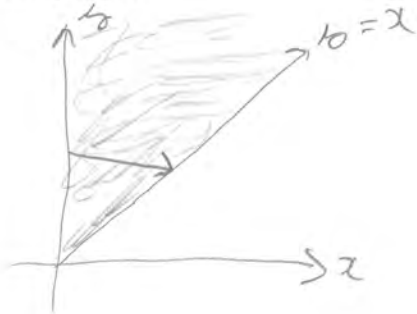
$$= \frac{1}{16} \int_0^2 x \left(\int_0^{x^2} y \, dy \right) dx = \frac{1}{16} \int_0^2 x \left. \frac{y^2}{2} \right|_0^{x^2} dx$$

$$= \frac{1}{32} \int_0^2 x^5 \, dx = \frac{1}{32} \left. \frac{x^6}{6} \right|_0^2 = \frac{2^6}{8 \cdot 4 \cdot 2 \cdot 3}$$

$$= \frac{\cancel{8} \cdot \cancel{8}}{\cancel{8} \cdot \cancel{8} \cdot 3} = \frac{1}{3}$$

10. Let $f_{x,y}(x,y) = \begin{cases} 4xye^{-(x^2+y^2)} & \text{for } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Find $P(Y > X)$. The answer is a number.



$$4\int_0^\infty \int_0^y xy\, e^{-x^2} e^{-y^2}\, dx\, dy$$

$u = -x^2 \quad du = -2x\,dx$

x	$u = -x^2$
y	$-y^2$
0	0

$$= 2\int_0^\infty y e^{-y^2} \int_0^y (-1) e^{-x^2} (-2x\,dx)\, dy$$

$$= 2\int_0^\infty y e^{-y^2} \left((-1) \int_0^{-y^2} e^u\, du \right) dy$$

$$= 2\int_0^\infty y e^{-y^2} (-1)\left(e^{-y^2} - 1\right) dy$$

$$= 2\int_0^\infty y e^{-y^2} \left(1 - e^{-y^2}\right) dy$$

$$= 2\int_0^{\infty} y e^{-y^2}\,dy - \frac{1}{2}\int_0^{\infty} y e^{-2y^2}\,4\,dy$$

$u = -y^2 \quad du = -2y\,dy \qquad u = -2y^2 \quad du = -4y\,dy$

y	
∞	$-\infty$
0	0

y	u
∞	$-\infty$
0	0

$$= (-1)\int_0^{-\infty} e^u\,du - \frac{1}{2}(-1)\int_0^{-\infty} e^u\,du$$

$$= \lim_{u \to -\infty} -e^u + 1 - \frac{1}{2}\left(\lim_{u \to \infty} -e^u + 1\right)$$

$$= 0 + 1 - \frac{1}{2}(0+1) = 1 - \frac{1}{2} = \frac{1}{2}$$