Sample Questions: Independence

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5R
15B

1. A jar contains 5 red balls and 15 black balls. Draw 2 balls randomly with replacement.

   (a) What is the probability that the first ball is red and the second is black? The answer is a number.

   $$P(R_1 \cap B_2) \overset{\text{ind}}{=} \frac{5}{20} \cdot \frac{15}{20} = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$$

   (b) What is the probability of one red and one black in any order? The answer is a number.

   $$P(R_1 \cap B_2) + P(B_1 \cap R_2) = \frac{6}{16}$$

2. Roll a fair die n times.

(a) What is the probability of observing at least one 4?

$$P(\text{At least one } 4) = 1 - P(\text{NO } 4\text{s})$$

$$= 1 - \left(\frac{5}{6}\right)^n$$

(b) How many times must you roll the die for the probability of at least one 4 to be 0.90 or more? The answer is a number.

$$1 - \left(\frac{5}{6}\right)^n \geq \frac{9}{10} \iff \frac{1}{10} \geq \left(\frac{5}{6}\right)^n$$

$$\iff \ln\left(\frac{1}{10}\right) \geq \ln\left(\left(\frac{5}{6}\right)^n\right) = n \ln\left(\frac{5}{6}\right)$$

$$\iff$$

$$\frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{5}{6}\right)} \leq n$$

$$\parallel$$

$$12.63$$

Say $n \geq 13$

3. A biased coin has $P(\text{Head}) = p$. Toss it three times.

   (a) List the elements of the sample space, along with their probabilities.

ω	$P\{\omega\}$
HHH	p^3
HHT	$p^2(1-p)$
HTH	$p^2(1-p)$
HTT	$p(1-p)^2$
THH	$p^2(1-p)$
THT	$p(1-p)^2$
TTH	$p(1-p)^2$
TTT	$(1-p)^3$

   (b) What is $P(\text{Two Heads})$?

$$3p^2(1-p)$$

4. It is clear from the last problem that the probability of a string with k heads is the same, regardless of their placement. Suppose we toss the biased coin n times. What is the probability of k heads (for $k = 0, \ldots, n$)?

$$\binom{n}{k} p^k (1-p)^{n-k}$$

5. Again, a biased coin has $P(\text{Head}) = p$. Toss it until the first head occurs, and then stop.

   (a) What is the probability that the first head appears on the fifth toss?

$$(1-p)^4 p$$

   (b) What is the probability that a head eventually occurs (on toss 1 or 2 or ...)?

$$P(H \text{ on toss } k) = (1-p)^{k-1} p$$

$$\sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^k$$

$$\frac{a^j}{1-a}$$

$$= \frac{p}{1-p} \cdot \frac{1-p}{1-(1-p)} = \frac{\cancel{p}}{\cancel{1-p}} \cdot \frac{\cancel{1-p}}{\cancel{p}}$$

$$= 1$$

(c) What is the probability that the first head occurs on an even numbered toss (toss 2 or 4 or …)? Simplify

$$(1-p)p + (1-p)^3 p + (1-p)^5 p + \cdots$$

$$\sum_{k=1}^{\infty} (1-p)^{2k-1} p = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^{2k}$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} \left[(1-p)^2\right]^k$$

$$= \frac{p}{\cancel{1-p}} \frac{(1-p)^{\cancel{2}}}{1-(1-p)^2} = \frac{p(1-p)}{1-(p^2-2p+1)}$$

$$= \frac{p(1-p)}{\cancel{1}-p^2+2p\cancel{-1}} = \frac{p(1-p)}{2p-p^2}$$

$$= \frac{\cancel{p}(1-p)}{\cancel{p}(2-p)} = \frac{1-p}{2-p}$$