

## Sample Questions: Independence

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1. A jar contains 5 red balls and 15 black balls. Draw 2 balls randomly with replacement.

(a) What is the probability that the first ball is red and the second is black? The answer is a number.

$$\begin{aligned} P(R_1 \cap B_2) &\stackrel{\text{ind}}{=} \frac{5}{20} \cdot \frac{15}{20} \\ &= \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16} \end{aligned}$$

(b) What is the probability of one red and one black in any order? The answer is a number.

$$\begin{aligned} P(R_1 \cap B_2) + P(B_1 \cap R_2) \\ = \frac{6}{16} \end{aligned}$$

2. Roll a fair die  $n$  times.

(a) What is the probability of observing at least one 4?

$$P(\text{At least one } 4) = 1 - P(\text{No } 4\text{'s}) \\ = 1 - \left(\frac{5}{6}\right)^n$$

(b) How many times must you roll the die for the probability of at least one 4 to be 0.90 or more? The answer is a number.

$$1 - \left(\frac{5}{6}\right)^n \geq \frac{9}{10} \Leftrightarrow \frac{1}{10} \geq \left(\frac{5}{6}\right)^n$$

$$\Leftrightarrow \ln\left(\frac{1}{10}\right) \geq \ln\left(\left(\frac{5}{6}\right)^n\right) = n \ln\left(\frac{5}{6}\right)$$

$\Leftrightarrow$

$$\frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{5}{6}\right)} \leq n \\ \text{"} \\ 12.63$$

Say  $n \geq 13$

3. A biased coin has  $P(\text{Head}) = p$ . Toss it three times.

(a) List the elements of the sample space, along with their probabilities.

$\omega$   $\{ \omega_1, \omega_2, \omega_3 \}$

$\omega$	Probability
HHH	$p^3$
• HHT	$p^2(1-p)$
• HTH	$p^2(1-p)$
HTT	$p(1-p)^2$
• THH	$p^2(1-p)$
THT	$p(1-p)^2$
TTH	$p(1-p)^2$
TTT	$(1-p)^3$

(b) What is  $P(\text{Two Heads})$ ?

$$3 p^2 (1-p)$$

4. It is clear from the last problem that the probability of a string with  $k$  heads is the same, regardless of their placement. Suppose we toss the biased coin  $n$  times. What is the probability of  $k$  heads (for  $k = 0, \dots, n$ )?

$$\binom{n}{k} p^k (1-p)^{n-k}$$

5. Again, a biased coin has  $P(\text{Head}) = p$ . Toss it until the first head occurs, and then stop.

(a) What is the probability that the first head appears on the fifth toss?

$$(1-p)^4 p$$

(b) What is the probability that a head eventually occurs (on toss 1 or 2 or ...)?

$$P(H \text{ on toss } k) = (1-p)^{k-1} p$$

$$\sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$\stackrel{\substack{a^j \\ 1-a}}{=} \frac{p}{1-p} \cdot \frac{1-p}{1-(1-p)} = \frac{p}{1-p} \cdot \frac{1-p}{p}$$

$$= 1$$

(c) What is the probability that the first head occurs on an even numbered toss (toss 2 or 4 or ...)? *Simplify*

$$(1-p)p + (1-p)^3p + (1-p)^5p + \dots$$

$$\sum_{k=1}^{\infty} (1-p)^{2k-1} p = \frac{p}{1-p} \sum_{k=1}^{\infty} (1-p)^{2k}$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} [(1-p)^2]^k$$

$$= \frac{p}{1-p} \frac{(1-p)^k}{1-(1-p)^2} = \frac{p(1-p)}{1-(p^2-2p+1)}$$

$$= \frac{p(1-p)}{\cancel{1-p^2} + 2p \cancel{-1}} = \frac{p(1-p)}{2p-p^2}$$

$$= \frac{p(1-p)}{p(2-p)} = \frac{1-p}{2-p}$$

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<http://www.utstat.toronto.edu/~brunner/oldclass/256f18>