Sample Questions: Counting Methods for Computing Probabilities

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1. Using the formula for $\binom{n}{r}$ from the formula sheet, and the Multiplication Principle, prove that the number of ways that n objects can be divided into r subsets with n_i objects in set i is $\binom{n}{n_1 \cdots n_r} = \frac{n!}{n_1! \cdots n_r!}$.

There are $\binom{n}{n_1}$ ways to choose the objects in set 1. For each of those, there are $\binom{n-n_1}{n_2}$ ways to choose objects in set 2. etc. By the Multiplication principle the total # of ways is

$$\binom{n}{n_1} \cdot \binom{n-n_1}{n_2} \cdot \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}} \cdot \frac{\cancel{(n-n_1-n_2)!}}{n_3!\,\cancel{(n-n_1-n_2-n_3)!}} \cdots \frac{\cancel{(n-n_1-\cdots-n_{r-1})!}}{n_r!\;0!}$$

$$= \frac{n!}{n_1!\,n_2! \cdots n_r!}$$

2. Sample r balls from a jar containing n numbered balls. How many outcomes are there is the sampling is

   (a) With replacement?

   $$n^r$$

   (b) Without replacement?

   $$\binom{n}{r}$$

3. Using the formula for ${}_nP_r$ from the formula sheet, and the Multiplication Principle, prove $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$.

Denote the number of unordered subsets by $\binom{n}{r}$. Place them in order. For each unordered subset, there are $r!$ ways to order the objects. By the Multiplication principle, there are

$$ {}_nP_r = \binom{n}{r} \cdot r! \text{ ordered subsets,} $$

$$ \Rightarrow \frac{n!}{(n-r)!} = \binom{n}{r} r! $$

$$ \Rightarrow \binom{n}{r} = \frac{n!}{r!\,(n-r)!} $$

∎

4. A jar contains 10 red balls and 20 blue balls. If 5 balls are randomly sampled without replacement, what is the probability of

(a) All blue?

$$\underbrace{\frac{\binom{20}{5}}{\binom{30}{5}}}_{\text{Good enough}} = \frac{\dfrac{20!}{\cancel{5!}\,15!}}{\dfrac{30!}{\cancel{5!}\,25!}} = \frac{20!\,25!}{15!\,30!} = \frac{2584}{23751}$$

$$\approx 0.109$$

(b) Two red and three blue?

$$\frac{\binom{10}{2}\cdot\binom{20}{3}}{\binom{30}{5}} = \frac{950}{2639} \approx .36$$

(c) At least one red?

$$1 - P(\text{All Blue})$$

$$= 1 - 0.109$$

$$= 0.891$$

(d) A jar contains 10 red balls and 20 blue balls. If 5 balls are randomly sampled without replacement, what is the probability of obtaining k red balls, $k = 0, \ldots, 5$?

Show some work. Don't simplify.

$$\frac{\binom{10}{k} \cdot \binom{20}{5-k}}{\binom{30}{5}}$$

5. A shipment of n electronic components has k defectives. If we sample m components without replacement, what is the probability of observing at least one defective?

P(At least one bad) = 1 − P(All good)

$$= 1 - \frac{\binom{n-k}{m}}{\binom{n}{m}}$$

✓

6. In how many ways can 20 basketball players be divided into 4 teams of 5?

$$\binom{20}{5\ 5\ 5\ 5} = \frac{20!}{(5!)^4} =$$

$$11,732,745,024$$

7. In how many ways can 6 red flags, 2 blue flags and 4 yellow flags be arranged? The flags are indistinguishable.

$$\binom{12}{6\ 2\ 4} = \frac{12!}{6!\,2!\,4!}$$

$$= 13,860$$

8. A standard deck of 52 cards has four "suits:" spades, diamonds, hearts and clubs. Within each suit, the face values of the 13 cards are 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. A "hand" of poker is 5 cards, selected randomly without replacement.

   (a) A "flush" is a hand with 5 cards all of the same suit. What is the probability of a flush?

$$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$$

   (b) A "straight" is a hand in which the 5 cards are in sequence. Suit is ignored. An Ace can be either high or low. What is the probability of a straight?

A 2 3 4 5 6 7 8 9 10 J Q K A

$$\frac{10 \cdot 4^5}{\binom{52}{5}}$$