Sample Questions: Continuous Random Variables

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1. The continuous random variable X has density $f(x) = \begin{cases} \frac{c}{x^{\alpha+1}} & \text{for } x \geq 1 \\ 0 & \text{for } x < 1 \end{cases}$

   where $\alpha > 0$.

   (a) Find the constant c

   $$1 = \int_1^\infty c\, x^{-\alpha-1}\, dx = \frac{c\, x^{-\alpha}}{-\alpha}\Big|_1^\infty$$

   $$= -\frac{c}{\alpha}\left(\lim_{x\to\infty} \frac{1}{x^\alpha} - 1\right) = -\frac{c}{\alpha}(0-1)$$

   $$= \frac{c}{\alpha} = 1 \text{ so } c = \alpha$$

   (b) Find the cumulative distribution function $F(x)$.

   For $x \geq 1$ $\quad F(x) = \int_1^x \alpha\, t^{-\alpha-1}\, dt$

   $$= \alpha \frac{t^{-\alpha}}{-\alpha}\Big|_1^x = (-1)\left(\frac{1}{x^\alpha} - \frac{1}{1^\alpha}\right) = 1 - \frac{1}{x^\alpha}, \text{ so}$$

   $$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 - \frac{1}{x^\alpha} & \text{for } x \geq 1 \end{cases}$$

   (c) The median of this distribution is that point m for which $P(X \leq m) = \frac{1}{2}$. What is the median? The answer is a function of α.

   $$F(m) = 1 - \frac{1}{m^\alpha} = \frac{1}{2} \Leftrightarrow \frac{1}{m^\alpha} = \frac{1}{2}$$

   $$\Leftrightarrow m^\alpha = 2 \Leftrightarrow m = 2^{1/\alpha}$$

2. Let $F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^\theta & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$

(a) If $\theta = 3$, what is $P\left(\frac{1}{2} < X \le 4\right)$? The answer is a number.

$$F(4) - F(\tfrac{1}{2}) = 1 - \frac{1}{2^\theta} = 1 - \left(\tfrac{1}{2}\right)^3 = 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

(b) Find $f(x)$.

For $0 < x < 1$, $\frac{d}{\partial x} x^\theta = \theta x^{\theta-1}$, so

$$f(x) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

3. If a random variable has density $f(x) = \frac{1}{2}e^{-|x|}$, find the cumulative distribution function.

If $x \le 0$, $F(x) = \int_{-\infty}^{x} \frac{1}{2} e^{t}\,dt = \frac{1}{2} e^{t} \Big|_{-\infty}^{x}$

$= \frac{1}{2}\left(e^{x} - \lim_{t \to -\infty} e^{t} \right) = \frac{1}{2}(e^{x} - 0)$

$= \frac{1}{2} e^{x}$

If $x > 0$ $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} \frac{1}{2} e^{-t}\,dt$

$= \frac{1}{2} + \frac{1}{2} \int_{0}^{-x} e^{u} (-1)\,du$ $\quad u = -t \quad du = -dt$

t	u
x	$-x$
0	0

$= \frac{1}{2} + \frac{1}{2} \int_{-x}^{0} e^{u}\,du = \frac{1}{2} + \frac{1}{2} e^{u} \Big|_{-x}^{0}$

$= \frac{1}{2} + \frac{1}{2}(1 - e^{-x})$

$= 1 - \frac{1}{2} e^{-x}$, so

$$F(x) = \begin{cases} \frac{1}{2} e^{x} & \text{for } x \le 0 \\ 1 - \frac{1}{2} e^{-x} & \text{for } x > 0 \end{cases}$$

4. The Uniform(a, b) distribution has density $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{Otherwise} \end{cases}$

Give the cumulative distribution function.

For $x \in [a, b]$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$$= \frac{1}{b-a}\, t \Big|_{a}^{x} = \frac{x-a}{b-a},$$

So

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

5. The Exponential(λ) distribution has density $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

(a) Show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$u = -\lambda x$

$du = -\lambda\,dx$

x	u
∞	$-\infty$
0	0

$$= \int_0^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= -\int_0^{-\infty} e^u\,du = \int_{-\infty}^{0} e^u\,du = e^u \Big|_{-\infty}^{0} = 1 - 0 = 1$$

(b) Find $F(x)$

For $x \geq 0$, $F(x) = \int_0^x \lambda e^{-\lambda t}\,dt$

$u = -\lambda t$

$du = -\lambda\,dt$

t	$u = -\lambda t$
x	$-\lambda x$
0	0

$$= -\int_0^{-\lambda x} e^u\,du = \int_{-\lambda x}^{0} e^u\,du$$

$$= e^u \Big|_{-\lambda x}^{0} = 1 - e^{-\lambda x}, \text{ so}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

(c) Still for the exponential density with $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$, prove the "memoryless" property:

$$P(X > t + s | X > s) = P(X > t)$$

for $t > 0$ and $s > 0$. For example, the probability that the conversation lasts at least t more minutes is the same as the probability of it lasting at least t minutes in the first place.

$$P(X > t + s | X > s) = \frac{P(X > t+s \cap X > s)}{P(X > s)}$$

$$= \frac{P(X > t+s)}{P(X > s)} = \frac{1 - F(t+s)}{1 - F(s)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = \frac{e^{-(\lambda t + \lambda s)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda t} \cancel{e^{-\lambda s}}}{\cancel{e^{-\lambda s}}} = e^{-\lambda t}$$

$$= P(X > t)$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$$

6. The Gamma(α, λ) distribution has density $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x}\, x^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$

   Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x}\, x^{\alpha-1}\, dx$$

$x = \frac{u}{\lambda}$

Let $u = \lambda x$ $\quad du = \lambda\, dx$

x	$u = \lambda x$
∞	∞
0	0

$$= \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda x}\, x^{\alpha-1}\, \underbrace{\lambda\, dx}_{du}$$

$$= \frac{\cancel{\lambda^{\alpha-1}}}{\Gamma(\alpha)} \int_0^\infty e^{-u} \left(\frac{u}{\cancel{\lambda}}\right)^{\alpha-1} du$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u}\, u^{\alpha-1}\, du$$

$$= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

7. The Normal(μ, σ) distribution has density $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{ \frac{(x-\mu)^2}{2\sigma^2} \right\}$

Let $X \sim \mathrm{N}(\mu, \sigma)$ and $Z = \frac{X-\mu}{\sigma}$. Find the density of ~~X~~. Z

$$f_z(z) = \frac{d}{dz} F_z(z) = \frac{d}{dz} P(Z \le z)$$

$$= \frac{d}{dz} P\left\{ \frac{X-\mu}{\sigma} \le z \right\} = \frac{d}{dz} P\{X - \mu \le \sigma z\}$$

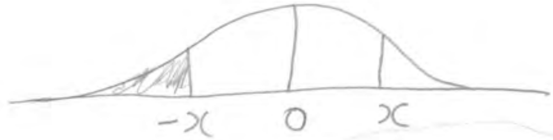
$$= \frac{d}{dz} P\{X \le \sigma z + \mu\} = \frac{d}{dz} F_x(\sigma z + \mu)$$

$$= f_x(\sigma z + \mu) \cdot \sigma$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\sigma z + \cancel{\mu} - \cancel{\mu})^2} \cdot \cancel{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(\cancel{\sigma} z)^2}{\cancel{\sigma^2}}}$$

$$\text{So } f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



8. Let $Z \sim N(0,1)$ (standard normal), so that $f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{z^2}{2}}$. If $x > 0$, show $F_z(-x) = 1 - F_z(x)$.

$$F_Z(-x) = (-1)\int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} (-1)\,dz$$

$z = -u$

$u = -z \quad du = -dz$

z	u
$-x$	x
$-\infty$	∞

$$= -\int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-u)^2}\,du$$

$$= \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\,du$$

$$= 1 - F_Z(x) \quad \blacksquare \quad \text{Because}$$

$$\underbrace{\int_{-\infty}^{x} f_Z(z)\,dz}_{F_Z(x)} + \int_{x}^{\infty} f_Z(z)\,dz = 1$$

$P(Z \leq x)$

9. Let $X \sim N(\mu = 50, \sigma = 10)$.

(a) Find $P(X < 60)$. The answer is a number.

$$P(X < 60) = P\left(\frac{X-\mu}{\sigma} < \frac{60-50}{10}\right)$$

$$= P(Z < 1) = 0.8413$$

(b) Find $P(X > 30)$. The answer is a number.

$$P(X > 30) = P\left(\frac{X-\mu}{\sigma} > \frac{30-50}{10}\right)$$

$$= P(Z > -2) = F(2) = 0.9772$$

(c) Find $P(30 < X < 55)$.

$$P(30 < X < 55) = P\left(\frac{30-50}{10} < \frac{X-\mu}{\sigma} < \frac{55-50}{10}\right)$$

$$= P\left(-2 < Z < \tfrac{1}{2}\right)$$

$$F_Z\left(\tfrac{1}{2}\right) - F_Z(-2)$$

$$= .6915 - (1 - 0.9772) = ?$$

8. The beta density with parameters α and β is $f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 \le x \le 1 \\ 0 & \text{Otherwise} \end{cases}$

Let $X \sim \text{Beta}(\alpha, \beta)$ with $\beta = 1$.

(a) Write the density of X for $0 \le x \le 1$. Simplify. You will prove $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ in homework.

For $0 \le x \le 1$ $f_x(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1}(1-x)^{1-1}$

$= \frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} x^{\alpha-1} = \alpha x^{\alpha-1}$

(b) Let $Y = 1/X$. For what values of y is $f_y(y) > 0$? Show some work.

If $0 < x < 1 \iff \infty > 1/x > 1$

$1 < y < \infty$

(c) Derive $f_y(y)$. Don't forget to specify where the density is greater than zero.

For $y \ge 1$ $f_y(y) = \frac{d}{dy} F_y(y) = \frac{d}{dy} P(Y \le y)$

$= \frac{d}{dy} P(1/X \le y) = \frac{d}{dy} P(X \ge \frac{1}{y})$

$= \frac{d}{dy}\left(1 - F_X(y^{-1})\right) = (-1) f_X(y^{-1})(-1) y^{-2}$

$= f_X(\frac{1}{y}) \cdot \frac{1}{y^2} = \alpha \left(\frac{1}{y}\right)^{\alpha-1} \cdot \frac{1}{y^2}$

$= \begin{cases} \frac{\alpha}{y^{\alpha+1}} & y \ge 1 \\ 0 & \text{otherwise} \end{cases}$

9. Let $Z \sim N(0,1)$ and $Y = Z^2$. $f_z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

(a) For what values of y is $f_y(y) > 0$?

$y > 0$

(b) Show that Y has a gamma distribution and give the parameters. You may use the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, without proof.

$$f_y(y) = \frac{d}{dy} P(Y \le y) = \frac{d}{dy} P(Z^2 \le y)$$

$$= \frac{d}{dy} P\{|Z| \le \sqrt{y}\} = \frac{d}{dy} P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= \frac{d}{dy}\left(F_z(\sqrt{y}) - F_z(-\sqrt{y})\right) \quad y^{\frac{1}{2}}$$

$$= f_z(\sqrt{y}) \cdot \frac{1}{2} y^{-\frac{1}{2}} + f_z(-\sqrt{y})(+1)\frac{1}{2} y^{-\frac{1}{2}}$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \left(f_z(\sqrt{y}) + f_z(-\sqrt{y})\right)$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y}\right)$$

$$= y^{-\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2}y}$$

$$= \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma(\frac{1}{2})} e^{-\frac{1}{2}y} \, y^{\frac{1}{2}-1}$$

Gamma $(\lambda = \frac{1}{2}, \alpha = \frac{1}{2})$

Compare $\frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha - 1}$

10. In this problem, the random variable X is transformed by its own distribution function. Let the continuous random vabriale X have distribution function $F_x(x)$, and let $Y = F_x(X)$.

over $\{x : f_x(x) > 0\}$

(a) For what values of y is $f_y(y) > 0$? Hint: as x ranges ~~from $-\infty$ to ∞~~, $F_x(x)$ ranges from $\underline{0}$ to $\underline{1}$.

(b) Find $f_y(y)$.

$$\frac{d}{dy} P(Y < y) = \frac{d}{dy} P(F_x(X) \leq y)$$

$$= \frac{d}{dy} P(F^{-1}(F(X)) \leq F^{-1}(y))$$

$$= \frac{d}{dy} P(X \leq F^{-1}(y))$$

$$= \frac{d}{dy} F_x(F_x^{-1}(y)) = \frac{d}{dy} y = 1$$

for $0 < y < 1$ uniform

$$f_y(y) = \begin{cases} 1 & \text{for } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
