Continuous Random Variables[1]

STA 256: Fall 2018

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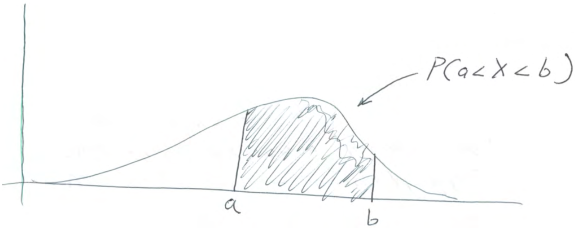
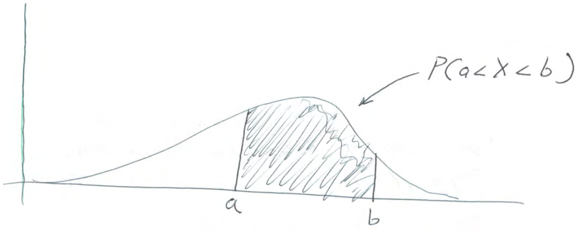
Continuous Random Variables: The idea

Probability is area under a curve

- Discrete random variables take on a finite or countably infinite number of values.
- Continuous random variables take on an *uncountably infinite* number of values.
- This implies that Ω is uncountable too, but we seldom talk about it.
- Probability is area under a curve — that is, area between a curve and the x axis.



The Probability Density Function



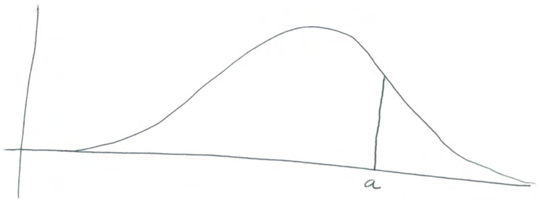
$$P(a < X < b) = \int_a^b f(x)\, dx$$

$f(x)$ is called the *density function* of X. Properties are

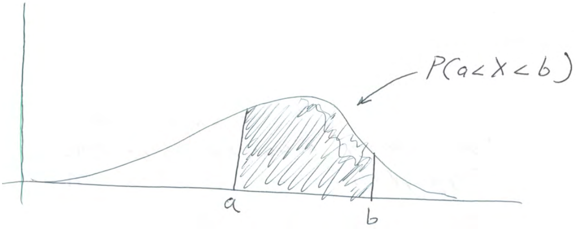
- $f(x) \geq 0$
- $f(x)$ is piecewise continuous.
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$

The probability of any individual value of X is zero

$$P(X = a) = 0$$



So
$P(< X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b).$



$$P(a < X < b) = F(b) - F(a)$$

$F'(x) = f(x)$

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= \int_{-\infty}^{x} f(t)\, dt \end{aligned}$$

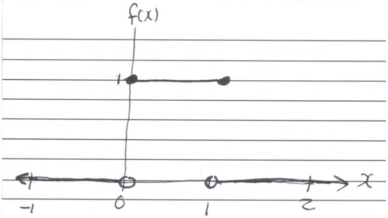
$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_{-\infty}^{x} f(t)\, dt = f(x)$$

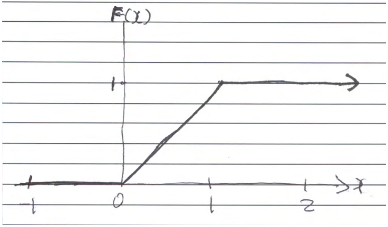
By the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

$F'(x) = f(x)$ is true for values of x where $F'(x)$ exists and $f(x)$ is continuous. For example, let

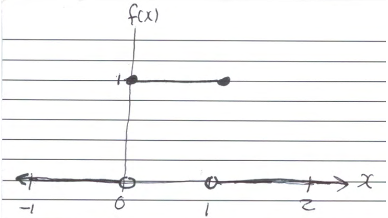
$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

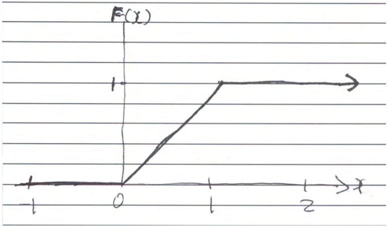




$F(x)$ is not differentiable at $x = 0$ and $x = 1$.

More comments



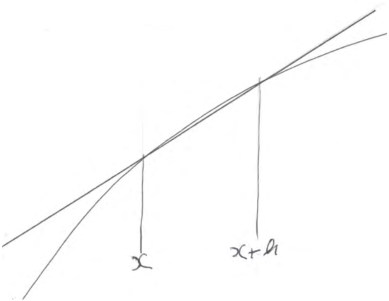


- $F(x)$ is not differentiable at $x = 0$ and $x = 1$.
- These are also the points where $f(x)$ is discontinuous.
- The exact value of $f(x)$ at those points cannot be recovered from $F(x)$.
- These are events of probability zero.
- They don't really affect anything.
- Recall that $f(x)$ is assumed piecewise continuous.
- The value of $f(x)$ at a point of discontinuity is essentially arbitrary. This causes no problems.

$f(x)$ is not a probability

$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$

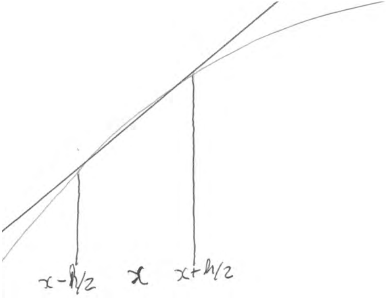
$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$



Another way to write $f(x)$

Instead of $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$

$$f(x) = \lim_{h\to 0} \frac{F(x+\frac{h}{2}) - F(x-\frac{h}{2})}{h}$$



Limiting slope is the same if it exists.

Interpretation

$$f(x) = \lim_{h \to 0} \frac{F(x + \frac{h}{2}) - F(x - \frac{h}{2})}{h}$$

- $F(x + \frac{h}{2}) - F(x - \frac{h}{2}) = P(x - \frac{h}{2} < X < x + \frac{h}{2})$
- So $f(x)$ is roughly proportional to the probability that X is in a tiny interval surrounding x.

Example

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Common questions:

- Prove it's a density.
- Find $F(x)$.

Prove it's a density

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

- Clearly $f(x) \geq 0$.
- It's continuous except at $x = 1$.
- Show $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Show $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx \\
&= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} 0\,dx \\
&= 0 + \int_{0}^{1} 2x\,dx + 0 \\
&= 2\frac{x^2}{2}\Big|_{0}^{1} \\
&= 1^2 - 0^2 = 1
\end{aligned}$$

Find $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$$f(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

There are 3 cases.

- If $x < 0$, $F(x) = \int_{-\infty}^{x} 0\, dt = 0$.
- If $0 \leq x \leq 1$,

$$F(x) = \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} 2t\, dt = x^2.$$

- If $x > 1$,

$$\begin{aligned} F(x) &= \int_{-\infty}^{0} 0\, dt + \int_{0}^{1} 2t\, dt + \int_{1}^{x} 0\, dt \\ &= 0 + 1 + 0 \\ &= 1 \end{aligned}$$

Putting the pieces together

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

The derivation does not need to be this detailed, but the final result has to be complete. More examples will be given.

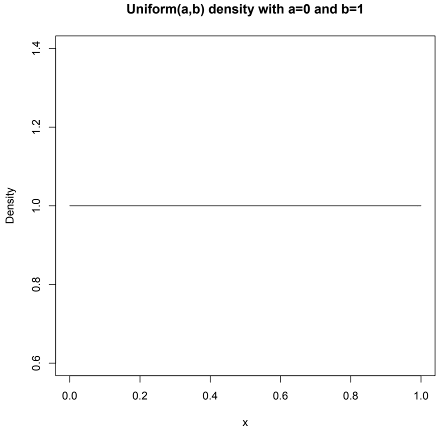
Common Continuous Distributions

- Uniform
- Exponential
- Gamma
- Normal
- Beta

The Uniform Distribution: $X \sim \text{Uniform}(a, b)$

Parameters $a < b$

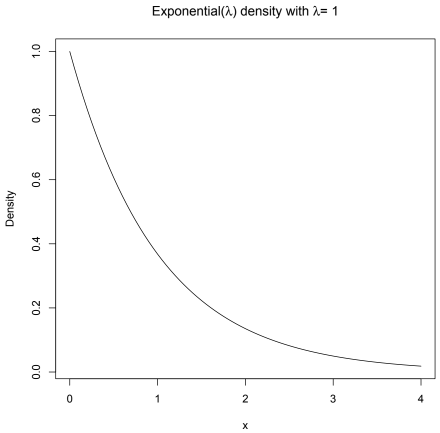
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{Otherwise} \end{cases}$$



The Exponential Distribution: $X \sim \text{Exponential}(\lambda)$

Parameter $\lambda > 0$

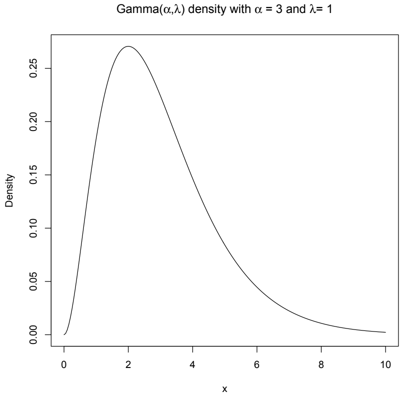
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$



The Gamma Distribution: $X \sim \text{Gamma}(\alpha, \lambda)$

Parameters $\alpha > 0$ and $\lambda > 0$

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} \, x^{\alpha-1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$



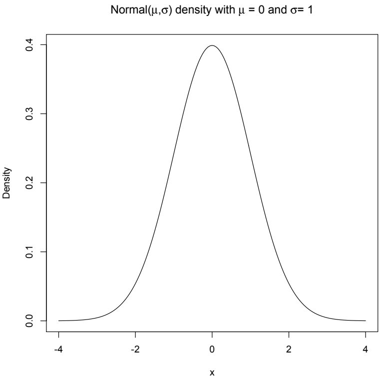
The gamma function is defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} \, t^{\alpha-1} \, dt$

Integration by parts shows $\Gamma(\alpha+1) = \alpha \, \Gamma(\alpha)$.

The Normal Distribution: $X \sim \mathrm{N}(\mu, \sigma)$

Parameters $\mu \in \mathbb{R}$ and $\sigma > 0$

$$
\begin{aligned}
f(x) &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp - \left\{ \frac{(x-\mu)^2}{2\sigma^2} \right\}
\end{aligned}
$$



The normal distribution is also called the Gaussian, or the "bell curve." if $\mu = 0$ and $\sigma = 1$, we write $X \sim \mathrm{N}(0,1)$ and call it the *standard normal.*

The Beta Distribution: $X \sim \text{Beta}(\alpha, \beta)$

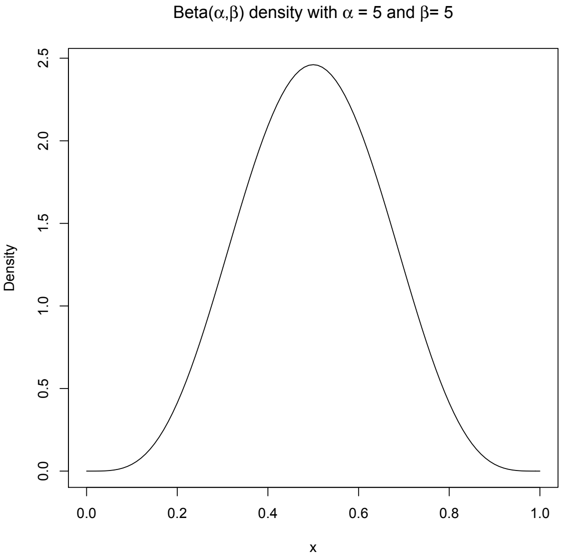
Parameters $\alpha > 0$ and $\beta > 0$

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

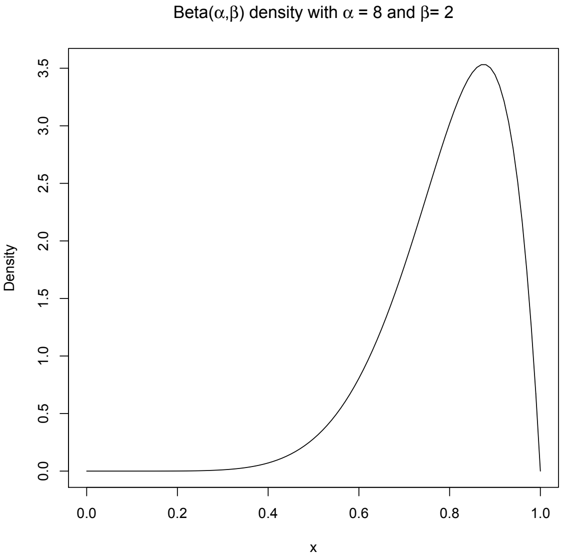
Using $\Gamma(n+1) = n\,\Gamma(n)$ and $\Gamma(\beta) = \int_0^\infty e^{-t}\, t^{\beta-1}\, dt$, note that a beta distribution with $\alpha = \beta = 1$ is Uniform(0,1).

The beta density can assume a variety of shapes, depending on the parameters α and β.

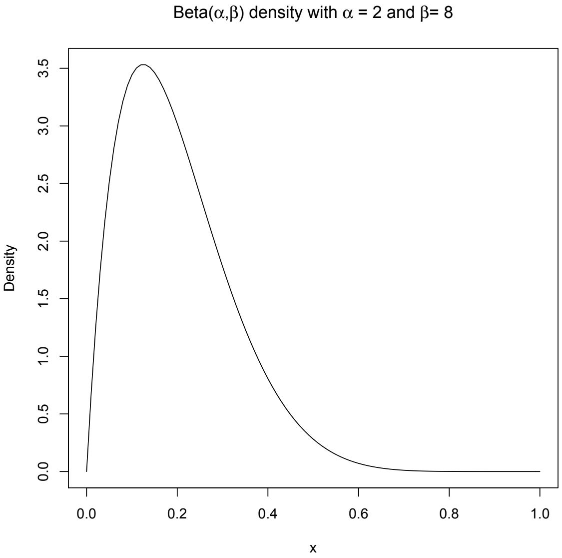
Beta density with $\alpha = 5$ and $\beta = 5$



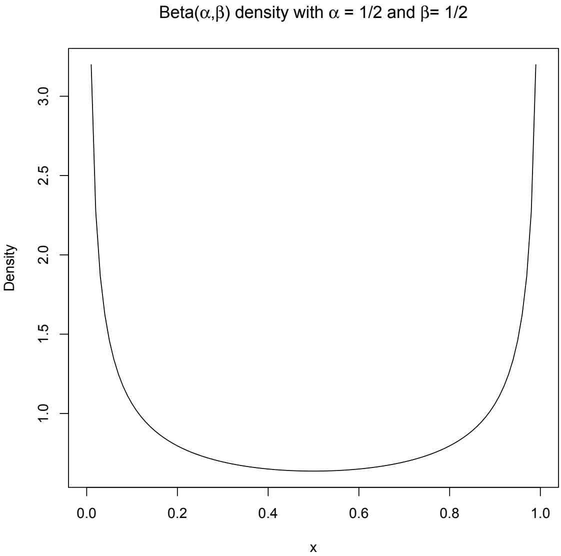
Beta density with $\alpha = 8$ and $\beta = 2$



Beta density with $\alpha = 2$ and $\beta = 8$



Beta density with $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$



Functions of a Random Variable

- Suppose you know the probability distribution of X.
- $Y = g(X)$.
- What is the probability distribution of Y?
- For example, X is miles per gallon.
- Y is litres per 100 kilometers for the same population of cars.
- You can make probability statements about X, but you need to make probability statements about Y

General approach to finding the density of $Y = g(X)$

First, find the set of y values where $f_y(y) > 0$.

- There are infinitely many right answers, differing only on a set of probability zero.
- If $f_x(x) > 0$, let $f_y(y) > 0$ for $y = g(x)$. This works.
- Then, for *one of those y values* (assuming $f_y(y)$ continuous there),

$$f_y(y) = \frac{d}{dy} F_y(y) = \frac{d}{dy} P(Y \leq y)$$

- Substitute for Y in terms of X.
- Try to solve for X.
- Express in terms of the cdf $F_x(x)$.
- Differentiate with respect to y.
- Usually use the chain rule.
- We need an example.

Example

Let $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y = 2X$. Find the density of Y.

$$f_x(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha - 1} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

First, where will $f_y(y)$ be non-zero?

- $f_x(x) > 0$ for $x \geq 0$.
- $x \geq 0 \Leftrightarrow y = 2x \geq 0$.
- So, for $y \geq 0$, ...

Derive the functional part of $f_y(y)$

$X \sim \text{Gamma}(\alpha, \lambda)$ and $Y = 2X$

$$
\begin{aligned}
f_y(y) &= \frac{d}{dy} F_y(y) \\
&= \frac{d}{dy} P(Y \leq y) \\
&= \frac{d}{dy} P(2X \leq y) \\
&= \frac{d}{dy} P\left(X \leq \frac{1}{2} y\right) \\
&= \frac{d}{dy} F_x\left(\frac{1}{2} y\right)
\end{aligned}
$$

$$
\begin{aligned}
&= f_x\left(\frac{1}{2} y\right) \cdot \frac{d}{dy} \frac{1}{2} y \\
&= \frac{1}{2} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} \exp\left\{-\lambda \frac{1}{2} y\right\} \left(\frac{1}{2} y\right)^{\alpha-1} \\
&= \frac{(\lambda/2)^\alpha}{\Gamma(\alpha)} \exp\left\{-\frac{\lambda}{2} y\right\} y^{\alpha-1}
\end{aligned}
$$

Compare gamma density: $f_x(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1}$ for $x \geq 0$.
Conclude $Y \sim \text{Gamma}(\alpha, \lambda/2)$.

Give the density of Y

Don't forget to specify where $f_y(y) > 0$

$$f_y(y) = \begin{cases} \frac{(\lambda/2)^\alpha}{\Gamma(\alpha)} \exp\left\{-\frac{\lambda}{2} y\right\} y^{\alpha-1}, & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

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