# Conditional Probability ${ }^{1}$ STA 256: Fall 2018 

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## Conditional Probability: The idea

- If event $B$ has occurred, maybe the probability of $A$ is different from the probability of $A$ overall.
- Maybe the chances of an auto insurance claim are different depending on the type of car.
- We will talk about the conditional probability of an insurance claim given that the car is a Dodge Charger.


## Restrict the sample space

To condition on the event $B$, make $B$ the new, restricted sample space.


Express the probability of $A$ as a fraction of the probability of $B$, provided the probability of $B$ is not zero.

$$
P(A \mid B)=\frac{P(A \cap B)}{P(B)}
$$

## Definition: The probability of $A$ given $B$

$$
\text { If } P(B)>0, P(A \mid B)=\frac{P(A \cap B)}{P(B)}
$$



$$
P(A \mid F)=\frac{P(A \cap F)}{P(F)}=\frac{0.01}{0.50}=0.02
$$

## Multiplication Law $P(A \mid B)=\frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B)=P(A \mid B) P(B)$

$$
P(A \cap B)=P(A \mid B) P(B)
$$

Useful for sequential experiments. A jar contains 15 red balls and 5 blue balls. What is the probability of randomly drawing a red and then a blue?
$P\left(R_{1} \cap B_{2}\right)=P\left(R_{1}\right) P\left(B_{2} \mid R_{1}\right)=\frac{15}{20} \frac{5}{19}=\frac{15}{76} \approx 0.197$

Make a Tree
Justified by the multiplication principle


## Trees

- Can be extended to more than 2 stages.
- Are best for small sequential experiments.
- Can allow you to side-step two important theorems, if the problem is set up nicely for you.
- The Law of Total Probability
- Bayes' Theorem.


## Law of Total Probability

Partition $\Omega$, into $B_{1}, B_{2}, \ldots B_{n}$, disjoint, with $P\left(B_{k}\right)>0$ for all $k$.


$$
\begin{aligned}
& A=\cup_{k=1}^{n}\left(A \cap B_{k}\right), \text { disjoint } \\
& \begin{aligned}
P(A) & =\sum_{k=1}^{n} P\left(A \cap B_{k}\right) \\
& =\sum_{k=1}^{n} P\left(A \mid B_{k}\right) P\left(B_{k}\right)
\end{aligned}
\end{aligned}
$$

## Example

Law of Total Probability: $P(A)=\sum_{k=1}^{n} P\left(A \mid B_{k}\right) P\left(B_{k}\right)$

In the 2 coins and one die example, we got $P(2)=\frac{7}{18}$ from a tree.

$$
\begin{aligned}
P(2) & =P(2 \mid \text { Coin } 1) P(\text { Coin } 1) \\
& +P(2 \mid \text { Coin } 2) P(\text { Coin } 2) \\
& +P(2 \mid \text { Die }) P(\text { Die }) \\
& =\frac{1}{2} \cdot \frac{1}{3}+\frac{1}{2} \cdot \frac{1}{3}+\frac{1}{6} \cdot \frac{1}{3} \\
& =\frac{7}{18}
\end{aligned}
$$

## Law of Total Probability

A general statement

Let $\Omega=\cup_{k=1}^{\infty} B_{k}$, disjoint, with $P\left(B_{k}\right)>0$ for all $k$. Then

$$
P(A)=\sum_{k=1}^{\infty} P\left(A \mid B_{k}\right) P\left(B_{k}\right)
$$

## Bayes Theorem: The idea

Bayes' Theorem allows you to turn conditional probability around, and obtain $P(B \mid A)$ from $P(A \mid B)$.

## Thomas Bayes (1701-1761)

Image from the Wikipedia


## Bayes' Theorem

## One version of many

Let $\Omega=\cup_{k=1}^{\infty} B_{k}$, disjoint, with $P\left(B_{k}\right)>0$ for all $k$. Then

$$
P\left(B_{j} \mid A\right)=\frac{P\left(A \mid B_{j}\right) P\left(B_{j}\right)}{\sum_{k=1}^{\infty} P\left(A \mid B_{k}\right) P\left(B_{k}\right)}
$$

An important special case is

$$
P(B \mid A)=\frac{P(A \mid B) P(B)}{P(A \mid B) P(B)+P\left(A \mid B^{c}\right) P\left(B^{c}\right)}
$$

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http://www.utstat.toronto.edu/~ ${ }^{\text {brunner/oldclass/256f18 }}$


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