Conditional Probability[1]

STA 256: Fall 2018

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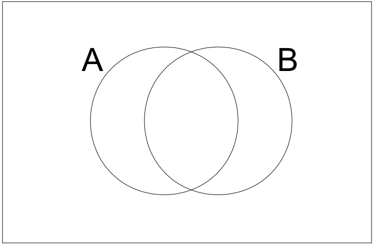
Conditional Probability: The idea

- If event B has occurred, maybe the probability of A is different from the probability of A overall.
- Maybe the chances of an auto insurance claim are different depending on the type of car.
- We will talk about the *conditional* probability of an insurance claim *given* that the car is a Dodge Charger.

Restrict the sample space

To condition on the event B, make B the new, restricted sample space.

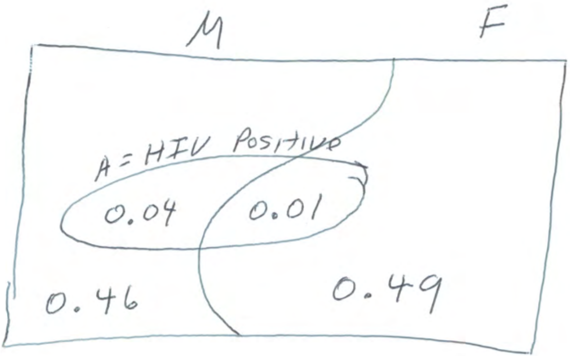


Express the probability of A as a fraction of the probability of B, provided the probability of B is not zero.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Definition: The probability of A given B

$$\text{If } P(B) > 0,\ P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$P(A|F) = \frac{P(A \cap F)}{P(F)} = \frac{0.01}{0.50} = 0.02$$

Multiplication Law

$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$

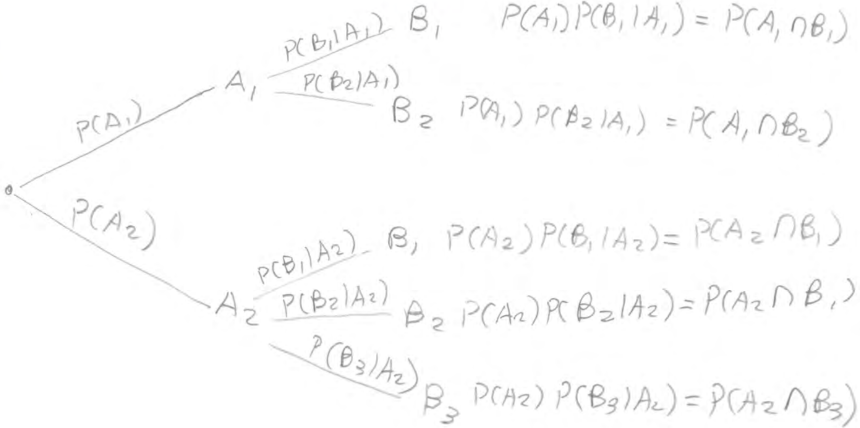
$$P(A \cap B) = P(A|B)P(B)$$

Useful for sequential experiments. A jar contains 15 red balls and 5 blue balls. What is the probability of randomly drawing a red and then a blue?

$P(R_1 \cap B_2) = P(R_1)P(B_2|R_1) = \frac{15}{20}\frac{5}{19} = \frac{15}{76} \approx 0.197$

Make a Tree

Justified by the multiplication principle

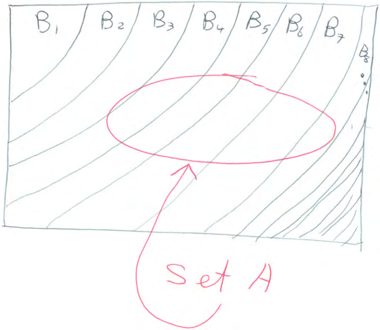


Trees

- Can be extended to more than 2 stages.
- Are best for *small* sequential experiments.
- Can allow you to side-step two important theorems, if the problem is set up nicely for you.
    - The Law of Total Probability
    - Bayes' Theorem.

Law of Total Probability

Partition Ω, into $B_1, B_2, \ldots B_n$, disjoint, with $P(B_k) > 0$ for all k.



$$A = \cup_{k=1}^{n} (A \cap B_k), \text{ disjoint}$$

$$\begin{aligned} P(A) &= \sum_{k=1}^{n} P(A \cap B_k) \\ &= \sum_{k=1}^{n} P(A|B_k) P(B_k) \end{aligned}$$

In the 2 coins and one die example, we got $P(2) = \frac{7}{18}$ from a tree.

$$\begin{aligned} P(2) &= P(2|\text{Coin } 1)\, P(\text{Coin } 1) \\ &+ P(2|\text{Coin } 2)\, P(\text{Coin } 2) \\ &+ P(2|\text{Die})\, P(\text{Die}) \\ &= \frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{6}\cdot\frac{1}{3} \\ &= \frac{7}{18} \end{aligned}$$

Law of Total Probability

A general statement

Let $\Omega = \cup_{k=1}^{\infty} B_k$, disjoint, with $P(B_k) > 0$ for all k. Then

$$P(A) = \sum_{k=1}^{\infty} P(A|B_k)P(B_k)$$

Bayes Theorem: The idea

Bayes' Theorem allows you to turn conditional probability around, and obtain $P(B|A)$ from $P(A|B)$.

Thomas Bayes (1701-1761)
Image from the Wikipedia

Let $\Omega = \cup_{k=1}^{\infty} B_k$, disjoint, with $P(B_k) > 0$ for all k. Then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{\infty} P(A|B_k)P(B_k)}$$

An important special case is

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

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