STA 256f18 Assignment Ten[1]

Please read Chapter 5 in the text. The following homework problems are not to be handed in. They are preparation for the final exam. All textbook problems are from Chapter Five. Use the formula sheet to do the problems. On tests and the final exam, you may use anything on the formula sheet unless you are being directly asked to prove or derive it.

1. For $n = 1, 2, \ldots$, let X_n have a beta distribution with $\alpha = n$ and $\beta = 1$.
   (a) Find the cumulative distribution function of X_n.
   (b) Show that X_n converges in probability to a constant, and find the constant. Convergence in probability means for all $\epsilon > 0$, $\lim_{n\to\infty} P\{|X_n - c| \geq \epsilon\} = 0$.
2. Use Chebyshev's inequality to prove the Law of Large Numbers.
3. The continuous mapping theorem for convergence in probability says that if $g(x)$ is a function that is continuous at $x = c$, and if T_n converges in probability to c, then $g(T_n)$ converges in probability to $g(c)$. A gamma random variable has expected value α/λ, something you could easily show if you needed to. Let $X_1, \ldots, X_n$ be a collection of independent gamma random variables with unknown parameter α, and known $\lambda = 6$. Find a random variable $T_n = g(\overline{X}_n)$ that converges in probability to α. The statistic T_n is a good way to estimate α from sample data.
4. Consider a *degenerate* random variable X, with $P(X = c) = 1$.
   (a) What is $F_x(x)$, the cumulative distribution function of X? Your answer must apply to all real x.
   (b) Give a formula for $M_x(t)$, the moment-generating function of X.
5. Recall that convergence of X_n to X in distribution means that $F_n(x) \to F(x)$ as $n \to \infty$ for all continuity points of $F(x)$. For the distribution of Question 1,
   (a) What is $\lim_{n\to\infty} F_{x_n}(x)$ for $x < 1$?
   (b) What is $\lim_{n\to\infty} F_{x_n}(x)$ for $x > 1$?
   (c) What do you conclude?
6. Sometimes, a sequence of random variables does not converge in distribution to anything. Let X_n have a continuous uniform distribution on $(0, n)$. Clearly, $\lim_{n\to\infty} F_{x_n}(x) = 0$ for $x \leq 0$. Find $\lim_{n\to\infty} F_{x_n}(x)$ for $x > 0$. Is $\lim_{n\to\infty} F_{x_n}(x)$ continuous? Is it a cumulative distribution function?

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7. Let X_n be a binomial (n, p_n) random variable with $p_n = \frac{\lambda}{n}$, so that $n \rightarrow \infty$ and $p \rightarrow 0$ in such a way that the value of $n\,p_n = \lambda$ remains fixed. Using moment-generating functions, find the limiting distribution of X_n.

8. Let $X_1, \ldots, X_n$ be independent geometric random variables, so that $E(X_i) = \frac{1}{p}$ and $Var(X_i) = \frac{1-p}{p^2}$. If $p = 1/2$ and $n = 64$, find the approximate $P(\overline{X}_n) > 2.5$. Answer: 0.0023

9. Let $X_1, \ldots, X_n$ be independent random variables from an unknown distribution with expected value 5.1 and standard deviation 4.8. Find the approximate probability that the sample mean will be greater than 6 for $n = 25$. Answer: 0.1736

10. The "normal approximation to the binomial" says that if $X \sim$ Binomial(n, p), then for large n,
$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$
may be treated as standard normal to obtain approximate probabilities. Where does this formula come from? Hint: What is the distribution of a sum of independent Bernoulli random variables?
