Math tools

Here are some useful theorems and definitions from advanced calculus and real analysis. Most of them are in the books on reserve in the math library.

Leibnitz's Rule Let $f(x,\theta)$ and $\frac{\partial f}{\partial \theta}$ be continuous in some region of the (x,θ) plane including $u_1 \le x \le u_2$ and $a \le \theta \le b$, and let $u_1(\theta)$ and $u_2(\theta)$ have continuous derivatives for $a \le \theta \le b$. Then

$$\frac{d}{d\theta}\int_{u_1(\theta)}^{u_2(\theta)} f(x,\theta)\,dx = \int_{u_1(\theta)}^{u_2(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\,dx + f(u_2(\theta),\theta)\frac{d}{d\theta}u_2(\theta) - f(u_1(\theta),\theta)\frac{d}{d\theta}u_1(\theta)$$

for $a \le \theta \le b$.

Taylor's Theorem Let the nth derivative $f^{(n)}$ be continuous in $[a,b]$ and differentiable in (a,b), with x and x_0 in (a,b). Then there exists a point ξ between x and x_0 such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \ldots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} \\ &+ \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!} \end{aligned}$$

where $R_n = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}$ is called the *remainder term*. If $R_n \to 0$ as $n \to \infty$, the resulting infinite series is called the *Taylor Series* for $f(x)$. There are other forms for the remainder term (with a different value of ξ) that sometimes prove useful.

Fubini's Theorem If a double integral (sum) converges absolutely, then the order of integration (summation) may be exchanged. If the quantity being integrated (or added up) is positive, then integration (summation) may always be exchanged, and if the result is ∞ in one direction, it is ∞ in the other direction too. See Fraser's *Probability and Statistics* or a measure theory book like Royden's *Real Analysis* for more precision.

Uniform Convergence of Sums Let $u_1(x), u_2(x), \ldots$ be a sequence of functions, and define $S_n(x) = \sum_{k=1}^{n} u_k(x)$. $\lim_{n\to\infty} S_n(s) = S(x)$ means $\forall\, \epsilon > 0$, $\exists\, N \ni$ if $n > N$, then $|S_n(x) - S(x)| < \epsilon$. In general, N will depend on x as well as ϵ. If N only depends on ϵ (often for all x in some interval) then we will say that $S_n(x)$ converges *uniformly* to $S(x)$ (again, often for the x in that interval).

Tests for Uniform Convergence of Sums

Weierstrass M test If there is a sequence of positive constants $M_1, M_2, \ldots$ such that $|u_n(x)| \leq M_n$ in some interval for each n, and if $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in that interval.

Dirichlet's test If the sequence of constants $a_n \downarrow 0$, and there exist constants N and P such that for $a \leq x \leq b$, $|S_n(x)| < P\ \forall\, n > N$, then the series $\sum_{n=1}^{\infty} a_n u_n(x)$ is uniformly convergent for $a \leq x \leq b$.

Theorems about Uniform Convergence of Sums

- If $u_n(x)$ is continuous in $[a, b]$ for $n = 1, 2, \ldots$, and if $S_n(x)$ converges uniformly to $S(x)$ in $[a, b]$, then $\lim_{x\to x_0} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \lim_{x\to x_0} u_n(x) = \sum_{n=1}^{\infty} u_n(x_0)$, where $x_0 \in [a, b]$, and right or left hand limits are used if x_0 is an endpoint of $[a, b]$.

- If $u_n(x)$ has a continuous derivative in $[a, b]$ for $n = 1, 2, \ldots$, and if $S_n(x)$ converges to $S(x)$ while $\sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$ converges uniformly in $[a, b]$, then $\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$.

Power Series A series of the form $\sum_{n=0}^{\infty} a_n (x-a)^n$ is called a *power series* in x. If it converges for $|x| < R$ and diverges for $|x| > R$, the constant R is called the *radius of convergence.*

A power series converges absolutely and uniformly in any interval that lies entirely within its interval of convergence, and hence may be differentiated term by term there. It may also be integrated term by term over any interval strictly within the interval of convergence.

Uniform Convergence of Integrals The integral $\int_a^\infty f(x,\theta)\,dx$ is said to converge uniformly for $\theta \in [\theta_1, \theta_2]$ if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{R} \ni$ if $u > N$, then $|\int_a^\infty f(x,\theta)\,dx - \int_a^u f(x,\theta)\,dx| < \epsilon$, where N depends on ϵ but not θ.

Tests for Uniform Convergence of Integrals

Weierstrass M test If there is a function $M(x) \geq 0$ such that $|f(x,\theta)| \leq M(x)$ for $\theta \in [\theta_1, \theta_2]$ and $x > a$, and if $\int_a^\infty f(x,\theta)\,dx$ converges, then $\int_a^\infty f(x,\theta)\,dx$ is uniformly and absolutely convergent for θ in $[\theta_1, \theta_2]$.

Dirichlet's test If the sequence of functions $\psi(x) \downarrow 0$ as $x \to \infty$, and there exists a constant P such that $|\int_a^u f(x,\theta)\,dx| < P$ for all $u > a$ and $\theta \in [\theta_1, \theta_2]$, then $\int_a^\infty f(x,\theta)\psi(x)\,dx$ is uniformly convergent for $\theta \in [\theta_1, \theta_2]$.

Theorems about Uniform Convergence of Integrals

- If $f(x,\theta)$ is continuous for $x \geq a$ and $\theta \in [\theta_1, \theta_2]$, and if $\int_a^\infty f(x,\theta)\,dx$ converges uniformly for $\theta \in [\theta_1, \theta_2]$, then

  $\lim_{\theta\to\theta_0} \int_a^\infty f(x,\theta)\,dx = \int_a^\infty \lim_{\theta\to\theta_0} f(x,\theta)\,dx = \int_a^\infty f(x,\theta_0)\,dx$, where $\theta_0 \in [\theta_1, \theta_2]$, and right or left hand limits are used if θ_0 is an endpoint of $[a, b]$.

- If $f(x,\theta)$ has a continuous derivative with respect to θ in $[\theta_1, \theta_2]$ for $x \geq a$ and if $\int_a^u f(x,\theta)\,dx$ converges to $\int_a^\infty f(x,\theta)\,dx$ while $\int_a^\infty \frac{\partial}{\partial\theta} f(x,\theta)\,dx$ converges uniformly in $[\theta_1, \theta_2]$, then $\frac{\partial}{\partial\theta}\int_a^\infty f(x,\theta)\,dx = \int_a^\infty \frac{\partial}{\partial\theta} f(x,\theta)\,dx$, provided a does not depend on θ. If it does, use Leibnitz's rule.