# Math Stat Review Assignment

**Part 1**: Answer each question T for true or F for false.

1. ____ Let P(A)>0, and $C_1$, $C_2$, ... be disjoint sets. Then $P(\bigcup_{n=1}^{\infty} C_n | A) = \sum_{n=1}^{\infty} P(C_n | A)$.

2. ____ X~Poisson(λ=2), and $Y=X^2$. $f_Y(3)=0$.

3. ____ If X has a binomial distribution, X is a discrete random variable.

4. ____ X~Normal(μ=5, $\sigma^2$=4). P(X>1)=.023.

5. ____ X~Binomial(n=20, p=.8). E(X)=3.2.

6. ____ Let $\mathbb{N}$ be the set of non−negative integers. If X~Normal(0,1), $P(X \in \mathbb{N})=0$.

7. ____ The moment−generating function of the random variable X is $M_X(t)=(1-2t)^{-1}$. X~Exponential(θ=2).

8. ____ The moment−generating function of the random variable X is $M_X(t)=(1-2t)^{-20}$. X~Chi−squared(r=20).

9. ____ If X has a Gamma distribution with parameters α and β, the density of X is symmetric about the mean αβ.

10. ____ The joint density of X and Y is $f_{XY}(x,y)=x+y$ for $0 \le x \le 1$ and $0 \le y \le 1$, zero otherwise. X and Y are independent.

11. ____ Let $X_1$, ..., $X_n$ be a random sample from a Normal $(\mu,\sigma^2)$ population. Then the joint density of the n random variables is $\frac{1}{\sigma^n (2\pi)^{n/2}} Exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\}$ .

12. ____ The moment−generating function of the random variable X is $M_X(t)=e^{5t+4t^2}$. X~Normal(μ=5, $\sigma^2$=8).

13. ____ You flip a fair coin 10 times. The variance of the number of heads is 5.

14. ____ X~Chi−squared(r=20). P(X<0)=0.

15. ____ X~Binomial(n=20, p=.8). $F_X(21)=0$.

16. ____ X~Gamma(α=2,β=20). Var(X)=40.

17. ____ If X has a Poisson distribution, $P(X \leq 0)=0$.

18. ____ The moment−generating function of the random variable X is $M_X(t)=(.2e^t+.8)^{20}$. X~Binomial(n=20, p=.8).

19. ____ If the events A and B are independent, $P(A|B)=P(B)$.

20. ____ X~Chi−squared(r=20). E(X)=40.

21. ____ Let X represent your weight yesterday, and Y represent your weight today. X and Y are dependent (not independent).

22. ____ X~Poisson($\mu$=1). $P(X=0)=e^{-1}$.

23. ____ X~Normal($\mu=0$, $\sigma^2=1$). $P(X>1)=P(X<-1)$.

24. ____ X~Poisson($\mu$=2). Var(X)=2.

25. ____ Let the random variables X and Y be independent. Then E(XY)=E(X)E(Y).

26. ____ X~Poisson($\mu$=2). $P(X=1)=2e^{-2}$.

27. ____ X~Gamma($\alpha=2,\beta=20$). P(X=1)=0.

28. ____ For two events A and B, if P(A)=0, it is possible for A and B to be both disjoint and independent.

29. ____ X~Exponential(o=2). $P(X=1)=e^{1/2}$.

30. ____ Let the random variables X and Y be independent. Then $P(X \leq x, Y \leq y) = F_X(x)F_Y(y)$.

31. ____ The moment−generating function of the random variable X is $M_X(t)=(1-2t)^{-20}$. X~Gamma($\alpha=2,\beta=20$).

32. ____ The moment−generating function of the random variable X is $M_X(t)=e^{2(e^t-1)}$ X~Poisson($\mu$=2).

33. ____ The moment−generating function of the random variable X is $M_X(t)=1$. This is impossible.

34. ____ X~Normal($\mu=0$, $\sigma^2=1$). $M_X(t)=e^{.5t^2}$.

35. ____ If $A \cap B = \bar{\varnothing}$, A and B are said to be independent.

36. ____ X~Poisson($\mu$=6.5). Var(X)=42.25.

37. ____ X~Exponential(θ=2). E(X)=2

38. ____ X~Normal($\mu$=5, $\sigma^2$=16). Var(X)=4.

39. ____ X~Normal($\mu$=5, $\sigma^2$=16). P(X>5)=1/2.

40. ____ Let $X_1, ..., X_n$ be a random sample from a Poisson distribution with parameter $\mu$. Then the joint density of the n random variables is $\frac{e^{-n\mu}\mu^{nx_i}}{nx_i!}$ , for $x_i$ = 0, 1, ... ; i= 1, ..., n; and zero otherwise.

# Review Assignment Part 2

1. Derive the means, variances and moment–generating functions in the table below. Obtain each mean and variance from the Moment–generating function, and also directly from the definition.

| Distribution | f(x) | $\mu$ | $\sigma^2$ | $M_X(t)$ |
|---|---|---|---|---|
| Bernoulli | $\theta^x(1-\theta)^{1-x}$ I{x=0,1} | $\theta$ | $\theta(1-\theta)$ | $\theta e^t + 1-\theta$ |
| Binomial | $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ I{x=0, ..., n} | $n\theta$ | $n\theta(1-\theta)$ | $(\theta e^t+1-\theta)^n$ |
| Poisson | $\frac{e^{-\lambda}\lambda^x}{x!}$ I{x=0, 1, ...} | $\lambda$ | $\lambda$ | $e^{\lambda(e^t-1)}$ |
| Exponential | $\frac{1}{\theta}e^{-x/\theta}$ I(x>0) | $\theta$ | $\theta^2$ | $(1-\theta t)^{-1}$ |
| Gamma | $\frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x/\beta}x^{\alpha-1}$ I(x>0) | $\alpha\beta$ | $\alpha\beta^2$ | $(1-\beta t)^{-\alpha}$ |
| Chi–square | $\frac{1}{2^{\nu/2}\Gamma(\nu/2)}e^{-x/2}x^{\nu/2-1}$ I(x>0) | $\nu$ | $2\nu$ | $(1-2t)^{-\nu/2}$ |
| Normal | $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\mu$ | $\sigma^2$ | $e^{\mu t+\frac{1}{2}\sigma^2t^2}$ |

2. Let the random variable X have an exponential distribution with mean $\theta$. Let $Y = \frac{2X}{\theta}$. Derive the probability density function $f_Y(y)$. Show your work.

3. Let the random variable X have a normal distribution with mean $\mu$ and variance $\sigma^2$. Let $Y = \frac{X-\mu}{\sigma}$. Derive the probability density function $f_Y(y)$. Show your work.

4. Let X have a normal distribution with mean $\mu$ and variance $\sigma^2$; that is, $X \sim N(\mu,\sigma^2)$. Let $Y = e^X$. (This means $X = \ln(Y)$, so Y has a log–normal distribution) Find the density $f_Y(y)$. Don't forget the support!

5. Let $X_1, \ldots, X_n$ be independent random variables with moment–generating functions $M_{X_i}(t)$, $i = 1, \ldots, n$. Let $Y = \sum_{i=1}^{n} X_i$. Starting from the definition of a moment–generating function and then using a convenient expression for $E[g(X_1, \ldots, X_n)]$, find $M_Y(t)$. Assume $X_1, \ldots, X_n$ are continuous, so you'll integrate. Show all your work.

6. Let $X_1, \ldots, X_n$ be independent Poisson random variables, all with the same parameter $\lambda>0$. Let $Y = \sum_{i=1}^{n} X_i$. Give $f_Y(y) = P(Y=y)$. Show your work. Remember, the support counts for half marks.

7. Let $X_1, \ldots, X_n$ be a random sample from a Normal $(0,\sigma^2)$ distribution.

   a) Let $Y_i = \frac{X_i^2}{\sigma^2}$. Use the distribution function technique to find the density of $Y_i$.

   b) Let $W = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2}$. Find the distribution of W. (That is, the distribution of W has a name. Name the distribution and give the value of the parameter.) Show your work.

# Review Assignment: Part 3

1. Let $X$ be a *continuous* random variable and let $a$ be a constant. Prove $E[a] = a$. You may use the "definition" $E[g(X)] = \int g(x)f(x)\,dx$.

2. Let $X$ and $Y$ be continuous random variables that are *independent.* Prove that $E[XY] = E[X]E[Y]$. Be very clear about where you are using the assumption of independence.

3. Let $X$ and $Y$ be *continuous* random variables. Prove $E[X+Y] = E[X]+E[Y]$. You may use the "definition" $E[g(X,Y)] = \int\int g(x,y)f(x,y)\,dx\,dy$, and you may exchange order of integration without comment. However, do *not* assume that $X$ and $Y$ are independent.

4. Let $X$ and $Y$ be random variables. Derive a formula for $Var(X+Y)$. You may use the formulas $Var(X) = E[X^2] - (E[X])^2$ and $Cov(X,Y) = E(XY) - E(X)E(Y)$ if you wish.

5. Let $f_{X,Y}(x,y) = e^{-x-y}I(x > 0)I(y > 0)$. What is $Cov(X,Y)$? There is a short quick way to do this problem, or you can do it the long way.

6. Let $X$ have a *Rayleigh* distribution. That is, $f_X(x) = 2\alpha x e^{-\alpha x^2} I(x > 0)$, and let $Y = \alpha X^2$.

   (a) (20 points) Find $f_Y(y)$. Make sure it is correct for all real $y$.

   (b) (5 points) Identify the distribution by name and give the values of its parameters.

7. Chebyshev's inequality states that for any random variable $X$ with $E(X) = \mu$ and $Var(X) = \sigma^2$ and for any $k > 0$, $P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$. Use this result to prove the following. Let $X_1, \ldots, X_n$ be a random sample from a population with expected value $\mu$ and variance $\sigma^2$. Then for all $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \epsilon) = 0.$$

8. The cumulative distribution function of the random variable $X$ is $F_X(x) = \frac{x+1}{2}I(-1 < x < 1) + I(x \geq 1)$.

(a) Find $P(-2 < X < 0)$

(b) Find $f_X(x)$. Make sure your answer is correct for all real $x$.

9. Let X have distribution function $F(x) = 1 - (1 + x)e^{-x}\, I(x > 0)$.

   **a.** Find $P(-4 < X < 3)$.

   **b.** Give a formula for the density function $f(x)$. Make sure it is correct for all real $x$.

10. Let the random variable $X$ satisfy $P(X = \mu) = 1$.

    (a) Derive the moment-generating function of $X$.

    (b) Sketch the cumulative distribution function of $X$.

11. The random variable $X$ has density $f_X(x) = 4e^{x-4e^x}\, I(-\infty < x < \infty)$. (The indicator is not really necessary but it may be helpful to you.) Find the density of $Y = e^X$. Make sure $f_Y(y)$ is correct for all real $y$.

12. Let $X$ have an exponential distribution with $\theta = 1$, and let $Y = X + 4$. Find the density of $Y$. The support is very important.

13. Let $U$ be distributed as $U(0, 1)$; that is, $f_U(u) = I(0 < u < 1)$. Let $Y = -\theta \log(1 - U)$, where $\theta > 0$, and of course it's the natural log. Find the density $f_Y(y)$; be sure to indicate where it's non-zero.

    (a) Use the Distribution Function Technique.

    (b) Use the Moment-generating Function Technique. You will "recognize" the answer.

    (c) Suppose you had a computer program that generated good pseudo-random numbers from a uniform distribution. How would you simulate values from the distribution of $Y$?

14. Let the distribution function F be differentiable and strictly increasing on $(-\infty, \infty)$, so that $F$ has a density $f$, and the inverse of the distribution function $F^{-1}$ exists. Let the random variable $U$ have density $f_U(u) = I(0 < u < 1)$, and let $Y = F^{-1}(U)$. Find the density $f_Y(y)$.

    (a) Use the Distribution Function Technique.

(b) Use the Moment-generating Function Technique. You will "recognize" the answer (or look at Part 1).

(c) Suppose you had a computer program that generated good pseudo-random numbers from a uniform distribution. How would you simulate values from the distribution of $Y$?

15. Let $X_1, \ldots, X_n$ be a random sample from a probability distribution with common distribution function $F(x)$ and density $f(x)$. Find the density of

    (a) $X_{(1)} = \text{Minumum}(X_1, \ldots, X_n)$

    (b) $X_{(n)} = \text{Maximum}(X_1, \ldots, X_n)$

16. The joint density of $X_1$ and $X_2$ is $f_{X_1,X_2}(x_1, x_2) = e^{-x_1-x_2} I(x_1 > 0) I(x_2 > 0)$. Find the density of $Y = X_1 + X_2$ any way you wish (more than one way will work). Make sure $f_Y(y)$ is correct for all real $y$.

17. Let $f_{X,Y}(x, y) = c\, xy I(x = 1, 2, 3) I(y = 1, \ldots, x)$

    (a) Find the constant $c$

    (b) Find $f_Y(y)$. You don't need to use indicator functions. Just give $f_Y(y)$ for the $y$ values with non-zero probability, and say "Zero for all other $y$."

    (c) Find $f_{X|Y}(x|2)$. You don't need to use indicator functions. Just give the values of $f_{X|Y}(x|2)$ for the $x$ values with non-zero probability, and say "Zero for all other $x$."

    (d) Find $E[X|Y = 2]$

18. Let $f_{X,Y}(x, y) = k\, xy^2 I(0 < x < 1) I(-x < y < x)$.

    (a) Sketch the support of $f_{X,Y}(x, y)$.

    (b) Find $k$.

    (c) Find $f_X(x)$. Make sure it is correct for all real $x$. Find $f_{Y|X}(y|x)$; your answer must apply to any $x$ between 0 and 1, and must be correct for all real $y$.

    (d) Find $E[Y|X = x]$, for some arbitrary fixed $x$ between 0 and 1.

19. Let $f_{X_1,X_2}(x_1, x_2) = \frac{1}{2} I(x_1 > 0) I(x_2 > 0) I(x_1 + x_2 < 2)$. Let $Y_1 = X_2 - X_1$ and $Y_2 = X_1$.

    (a) Sketch the support of $f_{X_1,X_2}(x_1, x_2)$.

    (b) Find $f_{Y_1,Y_2}(y_1, y_2)$. Make sure it is correct for all pairs of real numbers $(y_1, y_2)$.

    (c) (10 points) Sketch the support of $f_{Y_1,Y_2}(y_1, y_2)$.

20. Let $X_1, \ldots, X_n$ be a random sample from a *normal* population with $\mu = 0$ and $\sigma^2 = 1$, and let $Y = \sum_{i=1}^{n} X_i^2$. Find $f_Y(y)$. Make sure it is correct for all real $y$.

21. Let $X_1, \ldots, X_n$ be a random sample from an *exponential* population with parameter $\theta > 0$. Find the probability density function of the sample mean. Don't forget the support.

# Review Assignment: Part 4

1. Let the continuous random variables $X_1$ and $X_2$ be independent and normal, both with $\mu = 0$ and $\sigma^2 = 1$.

a. Find the joint density of $Y_1 = X_1/X_2$ and $Y_2 = X_2$. The ratio of two real numbers is real, so for once you don't need to worry about the support. The support of $Y_1$ and $Y_2$ is the whole plane $\mathbb{R}^2$.

b. Find the marginal density of $Y_1$. Again you don't need to indicate the support, because it's the whole real line. Hint: To get rid of the absolute value sign, split the integral at zero and realize that for $y_2 < 0$ , $|y_2| = \equiv y_2$.

2. Let $X_1$ and $X_2$ be independent N(0,1) random variables. Show that (contrary to what you might expect) $Y_1 = X_1+X_2$ and $Y_2 = X_1-X_2$ are independent random variables.

3. Let $X_1 \sim \text{Gamma}(\alpha_1,1)$ and $X_2 \sim \text{Gamma}(\alpha_2,1)$ be independent. Let $Y_1 = X_1+X_2$ and $Y_2 = \dfrac{X_1}{X_1 + X_2}$ .

a. Sketch the support of $Y_1$ and $Y_2$.

b. Derive the joint density of $Y_1$ and $Y_2$.

c. Find the marginal density of $Y_2$. Include an indicator function for the support.

4. Let $X_1 \sim \text{Normal}(0,1)$ and $X_2 \sim \text{Chi-square}(\nu)$ be independent. Find the density of $Y_1 = X_1/\sqrt{\frac{X_2}{\nu}}$ ; compare your result to the t density with $\nu$ degrees of freedom.

5. Let $X_1 \sim \text{Chi-square}(\nu_1)$ and $X_2 \sim \text{Chi-square}(\nu_2)$ be independent. Find the density of $Y_1 = \dfrac{X_1/\nu_1}{X_2/\nu_2}$ ; compare your result to the F density with $\nu_1$ and $\nu_2$ degrees of freedom.

6. Let $G_n$ have a Gamma distribution with parameters $\alpha$=n and $\beta$, where $\beta>0$ is not a function of n. Let $Y_n = G_n/n$ . Find the limiting distribution of $Y_n$. Hint: Find the limiting moment–generating function.