

Large sample tools¹

STA2101 Fall 2019

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Background Reading

Optional

- Section A.5 in Appendix A
- *An introduction to large sample theory* by Ferguson

Overview

- 1 Foundations
- 2 LLN
- 3 Consistency
- 4 CLT
- 5 Convergence of random vectors
- 6 Delta Method

Sample Space Ω , $\omega \in \Omega$

- Ω is a set, the underlying sample space.
- It could literally be the universe of websites from which we intend to sample.
- \mathcal{F} is a class of subsets of Ω .
- It could be the class of all subsets (if Ω is countable).
- There is a probability measure \mathcal{P} defined on the elements of \mathcal{F} .
- Maybe each website is equally likely to be chosen (with replacement).

Random variables are functions from Ω into the set of real numbers

$$Pr\{X \in B\} = Pr(\{\omega \in \Omega : X(\omega) \in B\})$$

Random Sample $X_1(\omega), \dots, X_n(\omega)$

- $T = T(X_1, \dots, X_n)$
- $T = T_n(\omega)$
- Let $n \rightarrow \infty$ to see what happens for large samples

Modes of Convergence

- Almost Sure Convergence
- Convergence in Probability
- Convergence in Distribution

Almost Sure Convergence

We say that T_n converges *almost surely* to T , and write $T_n \xrightarrow{a.s.} T$ if

$$Pr\{\omega : \lim_{n \rightarrow \infty} T_n(\omega) = T(\omega)\} = 1.$$

- Acts like an ordinary limit, except possibly on a set of probability zero.
- All the usual rules apply.
- Called convergence with probability one or sometimes strong convergence.
- In this course, convergence will usually be to a constant.

$$Pr\{\omega : \lim_{n \rightarrow \infty} T_n(\omega) = c\} = 1.$$

Strong Law of Large Numbers

Let X_1, \dots, X_n be independent with common expected value μ .

$$\overline{X}_n \xrightarrow{a.s.} E(X_i) = \mu$$

The only condition required for this to hold is the existence of the expected value.

Probability is long run relative frequency

- Statistical experiment: Probability of “success” is θ .
- Carry out the experiment many times independently.
- Code the results $X_i = 1$ if success, $X_i = 0$ for failure, $i = 1, 2, \dots$

Sample proportion of successes converges to the probability of success

Recall $X_i = 0$ or 1 .

$$\begin{aligned} E(X_i) &= \sum_{x=0}^1 x \Pr\{X_i = x\} \\ &= 0 \cdot (1 - \theta) + 1 \cdot \theta \\ &= \theta \end{aligned}$$

Relative frequency is

$$\frac{1}{n} \sum_{i=1}^n X_i = \bar{X}_n \xrightarrow{a.s.} \theta$$

Simulation

Using pseudo-random number generation by computer

- Estimate almost any probability that's hard to figure out
- Statistical power
- Weather model
- Performance of statistical methods
- Need confidence intervals for estimated probabilities.

Estimating power by simulation

Example: Bernoulli random sampling.

Recall the two test statistics for testing $H_0 : \theta = \theta_0$:

- $Z_1 = \frac{\sqrt{n}(\bar{Y} - \theta_0)}{\sqrt{\theta_0(1 - \theta_0)}}$
- $Z_2 = \frac{\sqrt{n}(\bar{Y} - \theta_0)}{\sqrt{\bar{Y}(1 - \bar{Y})}}$

When $\theta \neq \theta_0$, calculating $P\{|Z_2| > z_{\alpha/2}\}$ can be challenging.

Strategy for estimating power by simulation

- Generate a large number of random data sets under the alternative hypothesis.
- For each data set, test H_0 .
- Estimated power is the proportion of times H_0 is rejected.
- How accurate is the estimate?
- $\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{m}}$

Testing $H_0 : \theta = 0.50$ when true $\theta = 0.60$ and $n = 100$

$$Z_2 = \frac{\sqrt{n}(\bar{Y} - \theta_0)}{\sqrt{\bar{Y}(1 - \bar{Y})}}$$

```
> # Power by simulation
> set.seed(9999)
> m = 10000 # Monte Carlo sample size
> theta=0.60; theta0 = 1/2; n = 100
> Ybar = rbinom(m,size=n,prob=theta)/n # A vector of length m
> Z2 = sqrt(n)*(Ybar-theta0)/sqrt(Ybar*(1-Ybar)) # Another vector of length m
> power = length(Z2[abs(Z2>1.96)])/m; power
```

```
[1] 0.5394
```

Margin of error for estimated power

Confidence interval for an estimated probability was

$$\hat{p} \pm z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

```
# How about a 99 percent margin of error
> a = 0.005; z = qnorm(1-a)
> merror = z * sqrt(power*(1-power)/m); merror
[1] 0.0128391

> Lower = power - merror; Lower
[1] 0.5265609
> Upper = power + merror; Upper
[1] 0.5522391
```


Recall the Change of Variables formula: Let $Y = g(X)$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Or, for discrete random variables

$$E(Y) = \sum_y y p_Y(y) = \sum_x g(x) p_X(x)$$

This is actually a big theorem, not a definition.

Applying the change of variables formula

To approximate $E[g(X)]$

Simulate X_1, \dots, X_n from the distribution of X . Calculate

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n g(X_i) &= \frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{a.s.} E(Y) \\ &= E(g(X)) \end{aligned}$$

So for example

$$\frac{1}{n} \sum_{i=1}^n X_i^k \xrightarrow{a.s.} E(X^k)$$

$$\frac{1}{n} \sum_{i=1}^n U_i^2 V_i W_i^3 \xrightarrow{a.s.} E(U^2 V W^3)$$

That is, sample moments converge almost surely to population moments.

Approximate an integral: $\int_{-\infty}^{\infty} h(x) dx$

Where $h(x)$ is a nasty function.

Let $f(x)$ be a density with $f(x) > 0$ wherever $h(x) \neq 0$.

$$\begin{aligned}\int_{-\infty}^{\infty} h(x) dx &= \int_{-\infty}^{\infty} \frac{h(x)}{f(x)} f(x) dx \\ &= E \left[\frac{h(X)}{f(X)} \right] \\ &= E[g(X)],\end{aligned}$$

So

- Sample X_1, \dots, X_n from the distribution with density $f(x)$
- Calculate $Y_i = g(X_i) = \frac{h(X_i)}{f(X_i)}$ for $i = 1, \dots, n$
- Calculate $\bar{Y}_n \xrightarrow{a.s.} E[Y] = E[g(X)]$
- Confidence interval for $\mu = E[Y]$ is routine.

Convergence in Probability

We say that T_n converges *in probability* to T , and write $T_n \xrightarrow{P} T$ if for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\{\omega : |T_n(\omega) - T(\omega)| < \epsilon\} = 1$$

For us, convergence will usually be to a constant:

$$\lim_{n \rightarrow \infty} P\{|T_n - c| < \epsilon\} = 1$$

Convergence in probability (say to c) means no matter how small the interval around c , for large enough n (that is, for all $n > N_1$) the probability of getting that close to c is as close to one as you like.

We will seldom use the definition in this class.

Weak Law of Large Numbers

$$\overline{X}_n \xrightarrow{p} \mu$$

- Almost Sure Convergence implies Convergence in Probability
- Strong Law of Large Numbers implies Weak Law of Large Numbers

Consistency

$T = T(X_1, \dots, X_n)$ is a statistic estimating a parameter θ

The statistic T_n is said to be *consistent* for θ if $T_n \xrightarrow{P} \theta$ for all θ in the parameter space.

$$\lim_{n \rightarrow \infty} P\{|T_n - \theta| < \epsilon\} = 1$$

The statistic T_n is said to be *strongly consistent* for θ if $T_n \xrightarrow{a.s.} \theta$.

Strong consistency implies ordinary consistency.

Consistency is great but it's not enough.

- It means that as the sample size becomes indefinitely large, you probably get as close as you like to the truth.
- It's the least we can ask. Estimators that are not consistent are completely unacceptable for most purposes.

$$T_n \xrightarrow{a.s.} \theta \Rightarrow U_n = T_n + \frac{100,000,000}{n} \xrightarrow{a.s.} \theta$$

Consistency of the Sample Variance

$$\begin{aligned}\hat{\sigma}_n^2 &= \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2 \\ &= \frac{1}{n} \sum_{i=1}^n X_i^2 - \bar{X}^2\end{aligned}$$

By SLLN, $\bar{X}_n \xrightarrow{a.s.} \mu$ and $\frac{1}{n} \sum_{i=1}^n X_i^2 \xrightarrow{a.s.} E(X^2) = \sigma^2 + \mu^2$.

Because the function $g(x, y) = x - y^2$ is continuous,

$$\hat{\sigma}_n^2 = g\left(\frac{1}{n} \sum_{i=1}^n X_i^2, \bar{X}_n\right) \xrightarrow{a.s.} g(\sigma^2 + \mu^2, \mu) = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

Convergence in Distribution

Sometimes called *Weak Convergence*, or *Convergence in Law*

Denote the cumulative distribution functions of T_1, T_2, \dots by $F_1(t), F_2(t), \dots$ respectively, and denote the cumulative distribution function of T by $F(t)$.

We say that T_n converges *in distribution* to T , and write

$T_n \xrightarrow{d} T$ if for every point t at which F is continuous,

$$\lim_{n \rightarrow \infty} F_n(t) = F(t)$$

Again, we will seldom use this definition directly.

Univariate Central Limit Theorem

Let X_1, \dots, X_n be a random sample from a distribution with expected value μ and variance σ^2 . Then

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0, 1)$$

Connections among the Modes of Convergence

- $T_n \xrightarrow{a.s.} T \Rightarrow T_n \xrightarrow{p} T \Rightarrow T_n \xrightarrow{d} T.$
- If a is a constant, $T_n \xrightarrow{d} a \Rightarrow T_n \xrightarrow{p} a.$

Sometimes we say the distribution of the sample mean is approximately normal, or asymptotically normal.

- This is justified by the Central Limit Theorem.
- But it does *not* mean that \bar{X}_n converges in distribution to a normal random variable.
- The Law of Large Numbers says that \bar{X}_n converges almost surely (and in probability) to a constant, μ .
- So \bar{X}_n converges to μ in distribution as well.

Why would we say that for large n , the sample mean is approximately $N(\mu, \frac{\sigma^2}{n})$?

Have $Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0, 1)$.

$$\begin{aligned} Pr\{\bar{X}_n \leq x\} &= Pr\left\{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq \frac{\sqrt{n}(x - \mu)}{\sigma}\right\} \\ &= Pr\left\{Z_n \leq \frac{\sqrt{n}(x - \mu)}{\sigma}\right\} \approx \Phi\left(\frac{\sqrt{n}(x - \mu)}{\sigma}\right) \end{aligned}$$

Suppose Y is *exactly* $N(\mu, \frac{\sigma^2}{n})$:

$$\begin{aligned} Pr\{Y \leq x\} &= Pr\left\{\frac{\sqrt{n}(Y - \mu)}{\sigma} \leq \frac{\sqrt{n}(x - \mu)}{\sigma}\right\} \\ &= Pr\left\{Z_n \leq \frac{\sqrt{n}(x - \mu)}{\sigma}\right\} = \Phi\left(\frac{\sqrt{n}(x - \mu)}{\sigma}\right) \end{aligned}$$

Convergence of random vectors I

- ① Definitions (All quantities in boldface are vectors in \mathbb{R}^m unless otherwise stated)

★ $\mathbf{T}_n \xrightarrow{a.s.} \mathbf{T}$ means $P\{\omega : \lim_{n \rightarrow \infty} \mathbf{T}_n(\omega) = \mathbf{T}(\omega)\} = 1$.

★ $\mathbf{T}_n \xrightarrow{P} \mathbf{T}$ means $\forall \epsilon > 0, \lim_{n \rightarrow \infty} P\{\|\mathbf{T}_n - \mathbf{T}\| < \epsilon\} = 1$.

★ $\mathbf{T}_n \xrightarrow{d} \mathbf{T}$ means for every continuity point \mathbf{t} of $F_{\mathbf{T}}$,
 $\lim_{n \rightarrow \infty} F_{\mathbf{T}_n}(\mathbf{t}) = F_{\mathbf{T}}(\mathbf{t})$.

- ② $\mathbf{T}_n \xrightarrow{a.s.} \mathbf{T} \Rightarrow \mathbf{T}_n \xrightarrow{P} \mathbf{T} \Rightarrow \mathbf{T}_n \xrightarrow{d} \mathbf{T}$.

- ③ If \mathbf{a} is a vector of constants, $\mathbf{T}_n \xrightarrow{d} \mathbf{a} \Rightarrow \mathbf{T}_n \xrightarrow{P} \mathbf{a}$.

- ④ Strong Law of Large Numbers (SLLN): Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be independent and identically distributed random vectors with finite first moment, and let \mathbf{X} be a general random vector from the same distribution. Then $\bar{\mathbf{X}}_n \xrightarrow{a.s.} E(\mathbf{X})$.

- ⑤ Central Limit Theorem: Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. random vectors with expected value vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Then $\sqrt{n}(\bar{\mathbf{X}}_n - \boldsymbol{\mu})$ converges in distribution to a multivariate normal with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma}$.

Convergence of random vectors II

6 Slutsky Theorems for Convergence in Distribution:

- 1 If $\mathbf{T}_n \in \mathbb{R}^m$, $\mathbf{T}_n \xrightarrow{d} \mathbf{T}$ and if $f : \mathbb{R}^m \rightarrow \mathbb{R}^q$ (where $q \leq m$) is continuous except possibly on a set C with $P(\mathbf{T} \in C) = 0$, then $f(\mathbf{T}_n) \xrightarrow{d} f(\mathbf{T})$.
- 2 If $\mathbf{T}_n \xrightarrow{d} \mathbf{T}$ and $(\mathbf{T}_n - \mathbf{Y}_n) \xrightarrow{P} 0$, then $\mathbf{Y}_n \xrightarrow{d} \mathbf{T}$.
- 3 If $\mathbf{T}_n \in \mathbb{R}^d$, $\mathbf{Y}_n \in \mathbb{R}^k$, $\mathbf{T}_n \xrightarrow{d} \mathbf{T}$ and $\mathbf{Y}_n \xrightarrow{P} \mathbf{c}$, then

$$\begin{pmatrix} \mathbf{T}_n \\ \mathbf{Y}_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \mathbf{T} \\ \mathbf{c} \end{pmatrix}$$

Convergence of random vectors III

7 Slutsky Theorems for Convergence in Probability:

- 1 If $\mathbf{T}_n \in \mathbb{R}^m$, $\mathbf{T}_n \xrightarrow{P} \mathbf{T}$ and if $f : \mathbb{R}^m \rightarrow \mathbb{R}^q$ (where $q \leq m$) is continuous except possibly on a set C with $P(\mathbf{T} \in C) = 0$, then $f(\mathbf{T}_n) \xrightarrow{P} f(\mathbf{T})$.
- 2 If $\mathbf{T}_n \xrightarrow{P} \mathbf{T}$ and $(\mathbf{T}_n - \mathbf{Y}_n) \xrightarrow{P} \mathbf{0}$, then $\mathbf{Y}_n \xrightarrow{P} \mathbf{T}$.
- 3 If $\mathbf{T}_n \in \mathbb{R}^d$, $\mathbf{Y}_n \in \mathbb{R}^k$, $\mathbf{T}_n \xrightarrow{P} \mathbf{T}$ and $\mathbf{Y}_n \xrightarrow{P} \mathbf{Y}$, then

$$\begin{pmatrix} \mathbf{T}_n \\ \mathbf{Y}_n \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \mathbf{T} \\ \mathbf{Y} \end{pmatrix}$$

Convergence of random vectors IV

- 8 Delta Method (Theorem of Cramér, Ferguson p. 45): Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^k$ be such that the elements of $\dot{g}(\mathbf{x}) = \left[\frac{\partial g_i}{\partial x_j} \right]_{k \times d}$ are continuous in a neighborhood of $\boldsymbol{\theta} \in \mathbb{R}^d$. If \mathbf{T}_n is a sequence of d -dimensional random vectors such that $\sqrt{n}(\mathbf{T}_n - \boldsymbol{\theta}) \xrightarrow{d} \mathbf{T}$, then $\sqrt{n}(g(\mathbf{T}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \dot{g}(\boldsymbol{\theta})\mathbf{T}$. In particular, if $\sqrt{n}(\mathbf{T}_n - \boldsymbol{\theta}) \xrightarrow{d} \mathbf{T} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$, then $\sqrt{n}(g(\mathbf{T}_n) - g(\boldsymbol{\theta})) \xrightarrow{d} \mathbf{Y} \sim N(\mathbf{0}, \dot{g}(\boldsymbol{\theta})\boldsymbol{\Sigma}\dot{g}(\boldsymbol{\theta})')$.

An application of the Slutsky Theorems

- Let $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} ?(\mu, \sigma^2)$
- By CLT, $Y_n = \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Y \sim N(0, \sigma^2)$
- Let $\hat{\sigma}_n$ be *any* consistent estimator of σ .
- Then by 6.3, $\mathbf{T}_n = \begin{pmatrix} Y_n \\ \hat{\sigma}_n \end{pmatrix} \xrightarrow{d} \begin{pmatrix} Y \\ \sigma \end{pmatrix} = \mathbf{T}$
- The function $f(x, y) = x/y$ is continuous except if $y = 0$ so by 6.1,

$$f(\mathbf{T}_n) = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\hat{\sigma}_n} \xrightarrow{d} f(\mathbf{T}) = \frac{Y}{\sigma} \sim N(0, 1)$$

Univariate delta method

In the multivariate Delta Method 8, the matrix $\dot{g}(\boldsymbol{\theta})$ is a Jacobian. The univariate version of the delta method says that if $\sqrt{n}(T_n - \theta) \xrightarrow{d} T$ and $g''(x)$ is continuous in a neighbourhood of θ , then

$$\sqrt{n}(g(T_n) - g(\theta)) \xrightarrow{d} g'(\theta) T.$$

When using the Central Limit Theorem, *especially* if there is a $\theta \neq \mu$ in the model, it's safer to write

$$\sqrt{n}(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) T.$$

and then substitute for μ in terms of θ .

Example: Geometric distribution

Let X_1, \dots, X_n be a random sample from a distribution, with probability mass function $p(x|\theta) = \theta(1 - \theta)^{x-1}$ for $x = 1, 2, \dots$, where $0 < \theta < 1$.

So, $E(X_i) = \frac{1}{\theta}$ and $Var(X_i) = \frac{1-\theta}{\theta^2}$.

The maximum likelihood estimator of θ is $\hat{\theta} = \frac{1}{\bar{X}_n}$. Using the Central Limit Theorem and the delta method, find the approximate large-sample distribution of $\hat{\theta}$.

Solution: Geometric distribution

$$\mu = \frac{1}{\theta} \text{ and } \sigma^2 = \frac{1-\theta}{\theta^2}$$

CLT says $\sqrt{n} (\bar{X}_n - \mu) \xrightarrow{d} T \sim N(0, \frac{1-\theta}{\theta^2})$

Delta method says $\sqrt{n} (g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) T$.

$$g(x) = \frac{1}{x} = x^{-1}$$

$$g'(x) = -x^{-2}$$

So,

$$\begin{aligned} \sqrt{n} (g(\bar{X}_n) - g(\mu)) &= \sqrt{n} \left(\frac{1}{\bar{X}_n} - \frac{1}{\mu} \right) \\ &= \sqrt{n} (\hat{\theta} - \theta) \\ &\xrightarrow{d} g'(\mu) T = -\frac{1}{\mu^2} T \\ &= -\theta^2 T \sim N \left(0, \theta^4 \cdot \frac{1-\theta}{\theta^2} \right) \end{aligned}$$

Asymptotic distribution of $\hat{\theta} = \frac{1}{\bar{X}_n}$

Approximate large-sample distribution

Have $Y_n = \sqrt{n} (\hat{\theta} - \theta) \dot{\sim} N(0, \theta^2(1 - \theta))$.

So $\hat{\theta} = \frac{Y_n}{\sqrt{n}} + \theta = \hat{\theta} \dot{\sim} N\left(\theta, \frac{\theta^2(1-\theta)}{n}\right)$

We'll say that $\hat{\theta} = \frac{1}{\bar{X}_n}$ is approximately (or asymptotically) $N\left(\theta, \frac{\theta^2(1-\theta)}{n}\right)$.

Another example of $\sqrt{n} (g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) T$

Don't lose your head

Let $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} ?(\mu, \sigma^2)$

CLT says $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} T \sim N(0, \sigma^2)$

Let $g(x) = x^2$

Delta method says $\sqrt{n} (g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) T$.

So $\sqrt{n} (\bar{X}_n^2 - \mu^2) \xrightarrow{d} 2\mu T \sim N(0, 4\mu^2\sigma^2)$

Really? What if $\mu = 0$?

If $\mu = 0$ then $\sqrt{n} (\bar{X}_n^2 - \mu^2) = \sqrt{n} \bar{X}_n^2 \xrightarrow{d} 2\mu T = 0$

$\Rightarrow \sqrt{n} \bar{X}_n^2 \xrightarrow{p} 0$.

Already know from continuous mapping that $\bar{X}_n^2 \xrightarrow{p} \mu^2 = 0$.

Delta method reveals *faster convergence*.

Also ...

Have $\sqrt{n} \bar{X}_n^2 \xrightarrow{p} 0$. If we add another \sqrt{n} and if (say) $\sigma^2 = 1$ as well as $\mu = 0$,

$$n \bar{X}_n^2 = \left(\sqrt{n} (\bar{X}_n - \mu) \right)^2 \xrightarrow{d} Z^2 \sim \chi^2(1)$$

If $\sigma^2 \neq 1$, the target is Gamma($\alpha = \frac{1}{2}, \beta = 2\sigma$)

The delta method comes from Taylor's Theorem

Taylor's Theorem: Let the n th derivative $f^{(n)}$ be continuous in $[a, b]$ and differentiable in (a, b) , with x and x_0 in (a, b) . Then there exists a point ξ between x and x_0 such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \dots \\ &+ \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} + \frac{f^{(n+1)}(\xi)(x - x_0)^{n+1}}{(n+1)!} \end{aligned}$$

where $R_n = \frac{f^{(n+1)}(\xi)(x-x_0)^{n+1}}{(n+1)!}$ is called the *remainder term*. If $R_n \rightarrow 0$ as $n \rightarrow \infty$, the resulting infinite series is called the *Taylor Series* for $f(x)$.

Taylor's Theorem with two terms plus remainder

Very common in applications

Let $g(x)$ be a function for which $g''(x)$ is continuous in an open interval containing $x = \theta$. Then

$$g(x) = g(\theta) + g'(\theta)(x - \theta) + \frac{g''(\theta^*)(x - \theta)^2}{2!}$$

where θ^* is between x and θ .

Delta method

Using $g(x) = g(\theta) + g'(\theta)(x - \theta) + \frac{1}{2}g''(\theta^*)(x - \theta)^2$

Let $\sqrt{n}(T_n - \theta) \xrightarrow{d} T$ so that $T_n \xrightarrow{p} \theta$.

$$\begin{aligned}\sqrt{n}(g(T_n) - g(\theta)) &= \sqrt{n} \left(g(\theta) + g'(\theta)(T_n - \theta) + \frac{1}{2}g''(\theta_n^*)(T_n - \theta)^2 - g(\theta) \right) \\ &= \sqrt{n} \left(g'(\theta)(T_n - \theta) + \frac{1}{2}g''(\theta_n^*)(T_n - \theta)^2 \right) \\ &= g'(\theta) \sqrt{n}(T_n - \theta) \\ &\quad + \frac{1}{2}g''(\theta_n^*) \cdot \sqrt{n}(T_n - \theta) \cdot (T_n - \theta) \\ &\xrightarrow{d} g'(\theta)T + 0\end{aligned}$$

A variance-stabilizing transformation

An application of the delta method

- Because the Poisson process is such a good model, count data often have approximate Poisson distributions.
- Let $X_1, \dots, X_n \stackrel{i.i.d.}{\sim} \text{Poisson}(\lambda)$
- $E(X_i) = \text{Var}(X_i) = \lambda$
- $Z_n = \frac{\sqrt{n}(\bar{X}_n - \lambda)}{\sqrt{\bar{X}_n}} \xrightarrow{d} Z \sim N(0, 1)$
- Could say $\bar{X}_n \dot{\sim} N(\lambda, \lambda/n)$ and $\sum_{i=1}^n X_i \dot{\sim} N(n\lambda, n\lambda)$.
- Because the sum of independent Poissons is Poisson, this means Poisson-distributed variables with large λ are approximately normal.
- For analysis with normal linear models, approximate normality is good. Variance that depends on $E(Y_i)$ is not good.
- Can we fix it?

Variance-stabilizing transformation continued

- CLT says $\sqrt{n}(\bar{X}_n - \lambda) \xrightarrow{d} T \sim N(0, \lambda)$.
- Delta method says
$$\sqrt{n} (g(\bar{X}_n) - g(\lambda)) \xrightarrow{d} g'(\lambda) T = Y \sim N(0, g'(\lambda)^2 \lambda)$$
- If $g'(\lambda) = \frac{1}{\sqrt{\lambda}}$, then $Y \sim N(0, 1)$.

An elementary differential equation: $g'(x) = \frac{1}{\sqrt{x}}$

Solve by separation of variables

$$\frac{dg}{dx} = x^{-1/2}$$

$$\Rightarrow dg = x^{-1/2} dx$$

$$\Rightarrow \int dg = \int x^{-1/2} dx$$

$$\Rightarrow g(x) = \frac{x^{1/2}}{1/2} + c = 2x^{1/2} + c$$

We have found

$$\begin{aligned}\sqrt{n} (g(\bar{X}_n) - g(\lambda)) &= \sqrt{n} (2\bar{X}_n^{1/2} - 2\lambda^{1/2}) \\ &\xrightarrow{d} Z \sim N(0, 1)\end{aligned}$$

So,

- We could say that $\sqrt{\bar{X}_n}$ is asymptotically normal, with (asymptotic) mean $\sqrt{\lambda}$ and (asymptotic) variance $\frac{1}{4n}$.
- This calculation could justify a square root transformation for count data.
- Note that the transformation is increasing, so if Y_i is number of visitors to a website, $\sqrt{Y_i}$ could still be called “popularity.”

The arcsin-square root transformation

For proportions

Sometimes, variable values consist of proportions, one for each case.

- For example, cases could be high schools.
- The variable of interest is the proportion of students who enroll in university the year after graduation.
- This is an example of *aggregated data*.

The advice you sometimes get Still

When a proportion is the response variable in a regression, use the *arcsin square root* transformation.

That is, if the proportions are P_1, \dots, P_n , let

$$Y_i = \sin^{-1}(\sqrt{P_i})$$

and use the Y_i values in your regression.

Why?

It's a variance-stabilizing transformation (details omitted).

That was fun, but it was all univariate.

Because

- The multivariate CLT establishes convergence to a multivariate normal, and
- Vectors of MLEs are approximately multivariate normal for large samples, and
- The multivariate delta method can yield the asymptotic distribution of useful functions of the MLE vector,

We need to look at random vectors and the multivariate normal distribution.

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