What happens to $\frac{1}{n}SSE \approx MSE$ as $n \to \infty$? Noting that SSE is the same whether the model is centered or not, work with the centered model.

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i + \hat{y}_i)^2 = \sum_{i=1}^{n}(e_i + \hat{y}_i)^2$$

$$= \sum_{i=1}^{n} e_i^2 + 2\sum_{i=1}^{n}\hat{y}_i e_i + \sum_{i=1}^{n}\hat{y}_i^2 = SSE + 2\hat{y}^T e + \hat{y}^T\hat{y}$$

$$= SSE + 2(X\hat{\beta})^T e + \hat{y}^T\hat{y} = SSE + 2\hat{\beta}^T X^T e + (X\hat{\beta})^T X\hat{\beta}$$

$$= SSE + 0 + \hat{\beta}^T X^T X\hat{\beta} = SSE + \hat{\beta}^T \underbrace{X^T X (X^T X)^{-1}} X^T y = SSE + \hat{\beta}^T X^T y$$

So that $\frac{1}{n}\sum_{i=1}^{n} y_i^2 = \frac{1}{n}SSE + \frac{\cancel{n}}{n}\left(\hat{\Sigma}_x^{-1}\hat{\Sigma}_{xy}\right)^T \cancel{n}\hat{\Sigma}_{xy}$

$$= \frac{1}{n}SSE + \hat{\Sigma}_{xy}^T\,\hat{\Sigma}_x^{-1}\,\hat{\Sigma}_{xy}$$

Since y_i is really $y_i - \bar{y}$, $\frac{1}{n}\sum_{i=1}^{n} y_i^2 \xrightarrow{as} Var(y)$, and

$$\frac{1}{n}SSE = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \hat{\Sigma}_{xy}\,\hat{\Sigma}_x^{-1}\,\hat{\Sigma}_{xy} \xrightarrow{as} Var(y) - \Sigma_{xy}^T\,\Sigma_x^{-1}\,\Sigma_{xy}$$

Now if the model is correct, $Var(y) = Var(\beta^T x + \varepsilon) = \beta^T\Sigma_x\beta + \sigma^2$, and $\Sigma_{xy} = \Sigma_x\beta$. In that case

$$\frac{1}{n}SSE \xrightarrow{as} \beta^T\Sigma_x\beta + \sigma^2 - (\Sigma_x\beta)^T \underbrace{\Sigma_x^{-1}\Sigma_x}_{I}\beta$$

$$= \beta^T\Sigma_x\beta + \sigma^2 - \beta^T\Sigma_x\beta = \sigma^2$$

And MSE is consistent!

If the model is mis-specified, the important thing is that MSE approaches a constant as $n \to \infty$. Then, the estimated covariance matrix of $\hat{\beta}$

$$MSE(X^TX)^{-1} = MSE\left(n\tfrac{1}{n}X^TX\right)^{-1} = \tfrac{1}{n}MSE\left(\tfrac{1}{n}X^TX\right)^{-1} \xrightarrow{as} 0$$

And all the standard errors go to zero.